Problem 1. Let $f$ be bounded and holomorphic in $\{ z \in \mathbb{C} : |z| > R \}$.

(i) Show that $f$ has a Laurent series representation of the form

$$ f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots $$

containing non-positive powers of $z$ only.

(ii) If $M(r) = \sup_{|z|=r} |f(z)|$, show that $r \mapsto M(r)$ is strictly decreasing in $(R, +\infty)$ unless $f$ is constant.

Problem 2. Let $U$ be a bounded open connected set, $\{ f_n \}$ a sequence of functions holomorphic in $U$ and continuous in the closure of $U$. Assume that $\{ f_n \}$ converges uniformly on the boundary of $U$. Show that $\{ f_n \}$ converges uniformly on $U$.

Problem 3. Let $a_1, a_2, \ldots a_n$ be points on the unit circle. Prove that there exists a point $z$ on the unit circle so that the product of the distances from $z$ to $a_j$ is at least 1.

Problem 4. Suppose that $f(z)$ is holomorphic on the closed unit disk, and $|f(z)| \leq M$ for $|z| \leq 1$. Let $|f(0)| = a > 0$. Denote by $N$ the number of zeroes of $f$ in the disk $\{|z| < 1/3\}$. Show that

$$ N \leq \frac{1}{\log 2} \log \frac{M}{a}.$$

Hint: If $z_1, \ldots z_k$ are the zeroes of $f$ in $\{|z| < 1/3\}$, consider the function

$$ g(z) = \frac{f(z)}{\left(1 - \frac{z}{z_1}\right) \left(1 - \frac{z}{z_2}\right) \cdots \left(1 - \frac{z}{z_k}\right)}.$$

Problem 5. Determine the number of zeroes of the polynomial $2z^5 - 6z^2 + z + 1$ in the annulus $1 \leq |z| \leq 2$.

Problem 6. Let $\lambda$ be real, $\lambda > 1$. Show that the equation

$$ \lambda - z - e^{-z} = 0 $$

has exactly one root in the half-plane $\Re z \geq 0$, and that this root is real.

Problem 7. (i) Let $\{ f_n \}$ a sequence of functions which are all holomorphic and injective in the unit disk $\{|z| < 1\}$. Suppose that $\{ f_n \}$ converge uniformly in any smaller disk $\{|z| \leq r < 1\}$ to a non-constant holomorphic function $f$. Show that $f$ is also injective in the unit disk.

Remark. We will show very soon that a uniform limit of holomorphic functions is necessarily holomorphic, so the assumption that $f$ is holomorphic is actually extraneous.