**Problem Set 4**
due Tuesday, February 27

**Problem 1.** Prove the claim that we needed for the Maximum Modulus Principle:
Let $U \subset \mathbb{C}$ be open, connected, $f : U \rightarrow \mathbb{C}$ holomorphic. Suppose that all the values of $f$ lie on the unit circle, i.e. $|f(z)| = 1$ for all $z$. Then $f$ is constant.

**Problem 2.** Suppose $f$ and $g$ are holomorphic in a connected open set $U \subset \mathbb{C}$, and $|f(z)|^2 + |g(z)|^2 = 1$ for all $z \in U$. Show that $f$ and $g$ are constant.

**Problem 3.** A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is called doubly periodic if there are non-zero complex numbers $\alpha, \beta$, with $\alpha/\beta \notin \mathbb{R}$, such that $f(z + \alpha) = f(z + \beta) = f(z)$ for all $z \in \mathbb{C}$.
Show that every doubly periodic entire function is constant.

**Problem 4.** Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function, $N$ a positive integer, $C > 0$.
(a) Suppose that $|f(z)| \leq C|z|^N$ for all $z \in \mathbb{C}$. Show that $f$ is a polynomial.
(b) Suppose that $|f(z)| \geq C|z|^N$ for all $|z|$ large enough. Show that $f$ is a polynomial.

**Problem 5.** Consider the function
$$f(z) = 1 + z^2 + z^4 + z^8 + \ldots, \quad |z| < 1.$$ (Check that the radius of convergence is 1.)
Show that for every $a = e^{k2\pi i/2^n}$, $n \geq 0, k = 1, 2, \ldots 2^n-1$, $f(z)$ tends to $\infty$ when $z$ approaches $a$ along the radius of the circle, $\lim_{r \to 1^-} f(r \cdot a) = \infty$. Conclude that every point of the unit circle is a (non-isolated) singularity for $f$.

**Problem 6.** Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence equal to 1, and there are only poles of first order on the unit circle (no other singularities). Show that the sequence $\{a_n\}$ is bounded.

**Problem 7.** Suppose you want to represent the function
$$f(z) = \frac{1}{1 - z^2} + \frac{1}{3 - z}$$
by a Laurent series $\sum_{j=-\infty}^{\infty} a_j z^j$. How many such representations are there? In which region is each of them valid? Find the coefficients $a_j$ explicitly for each of these representations.