MAT 542 Complex Analysis I

Problem Set 3

due Tuesday, February 20

Problem 1. Consider an analytic function $g(z)$, given by a series $\sum a_n z^n$ with radius of convergence $R > 0$. Show that if the series

$$g(z) + g'(z) + g''(z) + \cdots + g^{(n)}(z) + \ldots \qquad (*)$$

converges at one single point $z_0 \in D_R(0)$, then in fact $R = \infty$, so $g(z)$ is analytic on $\mathbb{C}$, and the series (*) converges for every z. (It may be easier to assume that $z_0 = 0$ first.)

Moreover, show that in this case the series (*) converges uniformly on every bounded set in $\mathbb{C}$.

Problem 2. For $g(z)$ as in Problem 1, show that if the sequence

$$|g'(z)|, \quad \sqrt{|g''(z)|}, \quad \ldots, \quad \sqrt[n]{|g^{(n)}(z)|}, \ldots \qquad (**)$$

is bounded at one single point $z_0 \in D_R(0)$, then again $R = \infty$, so $g(z)$ is analytic on $\mathbb{C}$, and that sequence (**) stays bounded at every $z \in C$.

Moreover, show that in this case the sequence (**) has the same limit superior for all z.

Problem 3. Recall the Green formula from multivariable analysis:

$$\int_{\partial G} P\,dx + Q\,dy = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy;$$

here the functions $P = P(x, y)$ and $Q = Q(x, y)$ and their partial derivatives are all continuous in the closure $\overline{G}$ of the domain G, and ∂G stands for the counterclockwise-oriented boundary of G (we assume that ∂G is piecewise smooth).

Assume that the function $f : U \to C$ is holomorphic, and f' is continuous (using Cauchy's theorem, we will soon prove that the latter assumption holds for every holomorphic function). Also, assume that U is simply-connected, and γ is a piecewise smooth closed path with no self-intersections, so that γ bounds a domain $G \subset U$. Use the Green formula to show that

$$\int_\gamma f\,dz = 0,$$

thus giving an alternate proof for this (weaker) version of Cauchy's theorem.

In the above notation, what is $\int_\gamma \bar{z}\,dz$?

(Use the formula $\int f\,dz = \int (u + iv)(dx + idy)$; explain it if you can.)

Problem 4. Let $f : U \to \mathbb{C}$ be a holomorphic function. Adjust the version of the Goursat theorem that we proved in class to show that the integral of f over the boundary of any triangle contained in U is 0. (Some parts of the proof will be exactly the same as before, so you don't have to write down every detail; rather, streamline your proof and include the key pictures and estimates.)

Prove that the integral of f over the boundary of any (convex) polygon in U is 0.

Prove that the integral of f over any circle that bounds a disk in U is 0. (Approximate the circle by polygons; be careful with the $\gamma'(t)$ factor in the integral).

(Bonus) Can you generalize this argument to an arbitrary closed curve γ in a simply-connected domain U? You can assume that γ is smooth, as well as to make any other reasonable assumptions as needed.

Don't use Cauchy's theorem for this question; the goal is to get an alternate proof, at least for some special cases.