MAT 542 Complex Analysis I

Problem Set 2

due Tuesday, February 13

Problem 1. (i) Recall that we defined the *exponential map* via the power series $\exp(z) = \sum_{j=0}^{\infty} z^j/j!$ for all $z \in \mathbb{C}$. Show that $\exp(z+w) = \exp(z) \cdot \exp(w)$ for all $z, w \in \mathbb{C}$. (Use multiplication of power series from Problem Set 1.)

(ii) Show that given any $\zeta \in \mathbb{C}$ there exists a sequence $z_j \in \mathbb{C}$ with $|z_j| \to \infty$ such that $\exp(z_j) \to \zeta$. (Use any properties of $\exp(z)$.)

Problem 2. (i) Let U be the complex plane with the positive real axis deleted, and let $\log : U \to \mathbb{C}$ be defined by its principal value. Compute

(a) $\log(-1+i)$ (b) $\lim_{y \to 0}[\log(a+iy) - \log(a-iy)]$ for $a > 0$ and for $a < 0$.

(ii) Answer the same questions for the complex palne with the negative real axis deleted. Choose the branch of logarithm by taking $-\pi < \theta < \pi$.

Problem 3. Suppose that the function $f : U \to \mathbb{C}$ is holomorphic on an open set $U \subset \mathbb{C}$, U is connected, and $f'(z) = 0$ for all z. Show that f is a constant function.
(**Hint:** try looking at paths in U. You may use all the relevant facts from real analysis.)

Problem 4. For a continuous function $f : U \to \mathbb{C}$ and a piecewise smooth path $\gamma : [a, b] \to U$, we defined the integral $\int_\gamma f(z)dz$ as $\int_a^b f(\gamma(t))\gamma'(t)dt$.

Let P be a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$, choose $t_k \in [x_{k-1}, x_k]$, set $z_k = \gamma(x_k)$, $\xi_k = \gamma(t_k)$ for $1 \leq k \leq n$, and denote $||P|| = \max_{1 \leq k \leq n} |z_k - z_{k-1}|$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) - \int_\gamma f(z)dz| < \varepsilon$$

whenever $||P|| < \delta$, whatever the choice of x_k's and t_k's. (In fact, "the limit of Riemann sums" can be taken as the alternate definition of the integral.)

Problem 5. Let σ be a vertical segment parameterized by

$$\sigma(t) = z_0 + itc, \quad -1 \leq t \leq 1,$$

where z_0 is a fixed complex number, $c > 0$ is a fixed real number. (Draw the picture). Let $\alpha = z_0 + x$ and $\alpha' = z_0 - x$, where x is real positive. Find

$$\lim_{x \to 0} \int_\sigma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'} \right) dz.$$

(**Hint:** the answer is not 0.)

Problem 6. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk, $\partial\mathbb{D} = \{z : |z| = 1\}$ the unit circle parameterized by $\gamma(t) = e^{it}$ $(0 \leq t \leq 2\pi)$. Suppose $h : \partial\mathbb{D} \to \mathbb{C}$ is a continuous function. Show that

$$\int_\gamma \frac{h(\zeta)}{\zeta - z}\, d\zeta = 0 \text{ for all } z \in \mathbb{D} \text{ if and only if } \int_\gamma \zeta^j\, h(\zeta)\, d\zeta = 0 \text{ for all integers } j \leq -1.$$