MAT 540, Homework 7, due Wednesday, Nov 1 and Wednesday, Nov 8, in class

Questions 1, 2, and 3 are due on Nov 1. Please try to think about them by Monday, Oct 30, so that they could be discussed in class. Question 4, 5, 6 are due on Nov 8, although Question 4 is also good for the exam review.

1. Describe a simply connected space X and a covering space action of a group G such that X/G is the Klein bottle K.

This allows to compute $\pi_1(K) = G$ and to find $Deck\, X = G$, with explicit action of the elements of $Deck$.

2. Let X be the space obtained by attaching a disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$ to the circle $S = \{|z| = 1\}$ via the map $z \mapsto z^6$ from ∂D to S.

(a) What is the universal covering $\tilde{X}$ of X? Find the group of deck transformations $Deck\, \tilde{X}$. Describe explicitly the action of this group on the space $\tilde{X}$.

(b) Describe all path-connected coverings of X, up to isomorphism of coverings. You should give concrete explicit descriptions of spaces and covering maps, and prove your answer. Comment briefly on classification of basepointed coverings of X up basepoint-preserving isomorphism *vs.* classification up to isomorphism ignoring the basepoints.

(c) How does hierarchy of coverings works in this example?

3. Let X be the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus. Compute $\pi_1(X)$, describe the universal cover $\tilde{X}$ of X, and describe the action of $Deck\, \tilde{X}$ on $\tilde{X}$.

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4. Consider the "pseudocircle" $C = \{t, r, l, b\}$, with the topology given by the open sets

$$\{\{l, r, t, b\}, \{l, r, t\}, \{l, r, b\}, \{l, r\}, \{l\}, \{r\}, \emptyset\}.$$

Find the universal covering space, the fundamental group, and the higher homotopy groups of C.

Let $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ be the standard circle. Construct a continuous map $f : S^1 \to C$, such that $f_* : \pi_k(S^1) \to \pi_k(C)$ is an isomorphism for all $k > 0$. Is f a homotopy equivalence? Does your answer contradict Whitehead's theorem?

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5. Recall that $SO(3)$ is the group of orthogonal matrices of determinant 1. Geometrically. it is the group of rotations of R^3 about the origin.

(a) Prove that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$. The easiest way to do this is probably to use the fact that because each 3×3 matrix has a real eigenvalue, each rotation of $\mathbb{R}^3$ has a fixed axis (thus, it is represented as a rotation about some axis by some angle).

(b) The group $SU(2)$ is the group of unitary 2×2 matrices of determinant 1, with matrix multiplication. Elements of $SU(2)$ have the form

$$A = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix},$$

where α and β are complex numbers, $|\alpha|^2 + |\beta|^2 = 1$. As all linear groups, it is a topological space (matrix coefficients give an embedding into $\mathbb{C}^4$).

Prove that $SU(2)$ is homeomorphic to S^3. Moreover, prove that as a group, $SU(2)$ is isomorphic to the group of unit quaternions.

(c) Construct a group homomorphism $SU(2) \to SO(3)$ which is a 2-fold covering.

The group of quaternions consists of 8 elements $\{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication rules $ij = -ji = k$, etc (look them up somewhere). The division algebra

$$\mathbb{H} = \{a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, a, b, c, d \in \mathbb{R}\}$$

is a 4-dimensional vector space over $\mathbb{R}$. Unit quaternions correspond to $S^3 = \{a^2 + b^2 + c^2 + d^2 = 1\}$; this is a group with the multiplication induced from $\mathbb{H}$. I'm not sure if this material was discussed in other courses, but you only need to know the basic definitions (Wikipedia "Quaternion" article contains more than you need).

6. Find the fundamental group and the second homotopy group of the space S of all ellipsoids in $\mathbb{R}^3$ which have no equal semi-axes.

Hint: find a covering $p : SO(3) \to E$ and use it to solve the problem. Answer: the fundamental group of E is the group of quaternions.