MAT 540, Homework 4, due Wednesday, Oct 4, in class

1. Compute the following fundamental groups.

Your tools should be 1) finding convenient homotopy equivalences and 2) computing π_1 from cell attachments. Please don't use the van Kampen theorem. You can use the facts that $\pi_1(S^1) = \mathbb{Z}$ and that π_1(wedge of circles) is a free group on the corresponding number of generators.

(a) the complement of two disjoint lines in $\mathbb{R}^3$.

(b) the sphere S^n, $n \geq 2$, with k distinct points identified: if $K = \{x_1, x_2, \dots x_k\} \subset S^n$, what is $\pi_1(S^n/K)$?

(c) the quotient space X of S^2 under the identifications $x \sim -x$ for x in the equator S^1. (Note that X is *not* the same as $\mathbb{R}\mathrm{P}^2$: the opposite points outside the equator are *not* identified in pairs.)

(d) the CW complex Y obtained from S^1 by attaching two 2-cells, via maps $z \mapsto z^2$ and $z \mapsto z^3$, respectively. (Think of a 2-cell as unit disk $D \subset \mathbb{C}$, so that $\partial D = \{|z| = 1\}$).

(e) the space Z obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

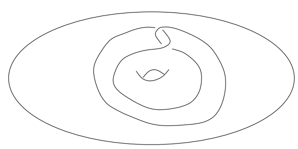
2. (a) Suppose that X retracts onto a subspace $A \subset X$. Let $i : A \hookrightarrow X$ be the inclusion map. Show that the induced homomorphism $i_* : \pi_k(A, x_0) \to \pi_k(X, x_0)$ is injective for every $k \geq 1$.

The standard application is that S^1 is not a retract of D^2, since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(D^2) = 0$; once you know that $\pi_n(S^n) = \mathbb{Z}$ (to be proved later), it similarly follows that S^n is not a retract of D^{n+1}.

Part (a) is directly useful for one of the questions below; for the other, you need a variant of the idea from (a).

(b) Prove that the Möbius band does not retract to its boundary circle.

(c) Prove that the solid torus $D^2 \times S^1$ does not retract to the curve $C \subset D^2 \times S^1$ shown in the figure below. (Think of $D^2 \times S^1$ as bounded in $\mathbb{R}^3$ by the 2-dimensional torus shown in the figure: the boundary surface of $D^2 \times S^1$ is the 2-torus. The curve is inside.)



3. Given an arbitrary group G, show that there exists a path-connected CW complex X with $\pi_1(X, x_0) = G$. Use the fact that every group is a quotient of a free group (possibly on infinitely many generators) by a normal subgroup.

4. Let K be the subspace of $\mathbb{R}^3$ formed by the Klein bottle intersecting itself in a circle, as shown in the figure. (Alternatively, K can be obtained by gluing two boundary circles of a cylinder to a third, nullhomotopic circle on the cylinder surface, with appropriate orientations.) Note that this is different from the usual Klein bottle.

(a) Compute $\pi_1(K)$.

(b) Does K retract to the circle C shown in the figure?

(c) Does K retract to any other circle?