MAT 540, Homework 3, due Wednesday, Sept 27, in class

You can use everything we proved in class, as well as the statements of all the problems from the previous homeworks.

1. Show that the following statements are equivalent. Each of them can be taken as a definition of an n-connected space X.

(1) For every $k \leq n$, any two continuous maps $S^k \to X$ are homotopic.

(2) For every $k \leq n$, any two continuous maps $(S^k, s_0) \to (X, x_0)$ are homotopic, i.e. $\pi_k(X) = 0$. Here, $s_0 \in S^k$ and $x_0 \in X$ are fixed basepoints.

(3) For every $k \leq n$, every continuous $g : S^k \to X$ can be extended to a continuous map $G : D^{k+1} \to X$, such that $G|_{S^k} = g$.

Strictly speaking, the statement about $\pi_k(X)$ is valid for $k \geq 1$; for $k = 0$, it is equivalent to path-connectedness.

2. Let's compute some fundamental groups. We haven't yet proved in class that $\pi_1(S^1, s_0) = \mathbb{Z}$, but you can use this fact in this question. You can also use the cellular approximation theorem (which implies, for instance, that $\pi_1(S^n, s_0) = 0$ for $n > 1$, as we saw in class). You should use various homotopy equivalences to reduce to spaces that you can easily understand. You *cannot* use the Van Kampen theorem in this question.

Compute the fundamental group of the following spaces.

(a) the complement of a line in $\mathbb{R}^3$;

(b) the complement of a line in $\mathbb{R}^4$;

(c) the complement of two disjoint lines in $\mathbb{R}^4$;

(d) the space $GL_+(2, \mathbb{R})$ of the real 2×2 matrices with positive determinant;

(e) the space $SL(2, \mathbb{R})$ of the real 2×2 matrices with determinant 1.

(f) the space X obtained from S^2 by attaching n 2-cells along an arbitrary collection of n circles in S^2.

Comments:

For (d) and (e), the topology on the matrix spaces is induced from the Euclidean spaces, as usual: $SL(2, \mathbb{R})$ and $GL_+(2, \mathbb{R})$ are subspaces of $\mathbb{R}^4$, via the matrix coefficients.

For (c): does the fundamental group depend on the choice of the two lines?

3. For any collection of spaces (X_α, x_α) with chosen basepoints, the *wedge sum* $\vee_\alpha X_\alpha$ is the quotient of the disjoint union $\sqcup_\alpha X_\alpha$ obtained by identifying all $x_\alpha \in X_\alpha$ to a single point.

(a) Suppose that X is a connected Hausdorff space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point. Show that X is homotopy equivalent to a wedge sum of S^1's and S^2's.

(As a first step, use a basic point-set topology argument to show that X is the quotient of the disjoint union of spheres by identifying the points corresponding to intersections.)

(b) Concretely, if X is the union of eight spheres centered at the vertices of a cube in $\mathbb{R}^3$ (with radius equal half the edge length of the cube), what is the homotopy type of X: how many 2-spheres and circles in the wedge sum?

4. (a) Prove that $\pi_n(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_n(X, x_0) \times \pi_n(Y, y_0)$, $n \geq 1$.

(b) Let $[f] \in \pi_1(X, x_0)$, $[g] \in \pi_1(Y, y_0)$, and consider the loops (f, y_0) and (x_0, g) in $(X \times Y, (x_0, y_0))$.

Since $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$, it follows that the elements $[(f, y_0)]$ and $[(x_0, g)]$ commute in $\pi_1(X \times Y, (x_0, y_0))$.

Find an explicit homotopy between the loops $(f, y_0) * (x_0, g)$ and $(x_0, g) * (f, y_0)$ in $(X \times Y, (x_0, y_0))$.