MAT 540, Homework 10, due Friday, Dec 8

1. Let $S^n = \{x_1, x_2, \dots, x_{n+1} \,|x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$ be the sphere in $\mathbb{R}^{n+1}$.
(a) Compute the degree of a reflection map, $(x_1, x_2, \dots x_n) \to (-x_1, x_2, \dots, x_n)$.

(b) Compute the degree of the reflection map, $(x_1, x_2, \dots x_n) \to (-x_1, -x_2, \dots, -x_n)$.

(c) For every $m \in \mathbb{Z}$, explain how to construct a map $S^n \to S^n$ of degree m.

(d) Prove that if $f : S^n \to S^n$ be a continuous map without fixed points, then f has degree $(-1)^{n+1}$.

2. Let G be a group acting freely on S^n for some even n. (This means that every nontrivial element $g \in G$ is a homeomorphism $g : S^n \to S^n$ without fixed points, and the product operation in G is given by composition of maps of S^n.) Prove that G is isomorphic to $\mathbb{Z}/2$.

3. This long question defines the winding number of a loop around a point and establishes its properties.

Suppose $u : S^1 \to \mathbb{R}^2$ is a continuous map, and $x \notin u(S^1)$. Then u determines an element in $\pi_1(\mathbb{R}^2 - \{x\}) = \mathbb{Z}$, called **the winding number of u with respect to x**, and often denoted $\operatorname{ind}_x u$. Note that for each $x \in \mathbb{R}^2$, we will choose the homotopy class of the *couterclockwise* standard loop going once around x as the generator $1 \in \mathbb{Z}$

(a) Draw a loop with some self-intersections, pick a point in each connected component of the complement of your loop, and compute the corresponding winding numbers.

(b) Show that the winding number $\operatorname{ind}_x u$ can be characterized as the degree of the map

$$\phi_{u,x} : S^1 \to S^1, \qquad \phi_{u,x}(z) = \frac{u(z) - x}{|u(x) - z|}.$$

(c) Prove that the formula $x \mapsto \operatorname{ind}_x u$ defines a locally constant function on $\mathbb{R}^2 - u(S^1)$. (It follows that if u is a "nice" curve, possibly with some self-intersections, so that it divides $\mathbb{R}^2$ into some connected components, and the winding number remains the same within each component.)

(d) Let $u : S^1 \to \mathbb{R}^2$, and suppose that $x, y \in \mathbb{R}^2 - u(S^1)$, such that $\operatorname{ind}_x u \neq \operatorname{ind}_y u$. Show that any path from x to y must intersect $u(S^1)$.

(e) Show that if $u(S^1)$ is contained in a disk D and $x \notin D$, then $\operatorname{ind}_x u = 0$.

(f) If $u, v : S^1 \to \mathbb{R}^2$ are two loops with common basepoint $u(s_0) = v(s_0)$, and uv is their product, then

$$\operatorname{ind}_x uv = \operatorname{ind}_x u + \operatorname{ind}_x v \text{ for every } x \notin uv(S^1).$$

(g) Let R be a ray in $\mathbb{R}^2$ starting at x. Show that R meets $u(S^1)$ in at least $|\operatorname{ind}_x u|$ points.

Please also do questions 32, 33 from Hatcher Section 4.2.