MAT 539 Algebraic Topology

Problem Set 3

due Wednesday, September 21

Please do Question 9 p.156 (use cellular homology) and Question 17 p.157 from Hatcher.

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Problem 1. Let X be a CW-complex, A its subcomplex. Prove that A has a neighborhood U in X such that U deformation retracts onto A.

(This was a key property that together with the excision theorem allowed us to show, for example, that $H_*(X_n, X_{n-1}) = H_*(X_n/X_{n-1})$, which was useful because the latter quotient space is a bouquet of spheres.)

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Problem 2. All CW-complexes in this problem are supposed to be finite.

(a) Show that the Euler characteristic of a CW-complex X can be computed as

$$\chi(X) = \sum (-1)^k c_k,$$

where c_k is the number of k-cells.

(b) Suppose that X and Y are CW-complexes. Show that the $X \times Y$ is a CW-complex, and

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

(c) If a CW-complex X is the union of subcomplexes A and B, show that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

(Note that $A \cap B$ is also a CW subcomplex of X – why?)

(d) Let X be a CW complex, $\tilde{X} \to X$ an n-fold covering. Explain why $\tilde{X}$ is also a CW-complex, and show that $\chi(\tilde{X}) = n\chi(X)$.

Problem 3. Later on, the Künneth formula will tell us how to find the singular homology of a product $X \times Y$ in terms of the homology of X and Y. When all homology groups of Y are free abelian, it says

$$H_n(X \times Y) = \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y).$$

For the moment, prove the special case from Question 36 p. 158 (you will need to read about the relative Mayer–Vietoris sequence in Hatcher).

Compute the singular homology of the n-dimensional torus $T^n = (S^1)^n$.

Problem 4. (i) Think of the n-torus T^n as the cube I^n with opposite sides identified. Consider the CW-structure on T^n obtained from the faces of the cube (of any dimension). This CW-structure has $\binom{n}{k}$ k-cells for each k. Calculate the cellular homology by showing that the boundary maps in the cellular chain complex are zero.

(ii) Now, think of T^n as $\mathbb{R}^n/\mathbb{Z}^n$. Let A be a linear transformation of $\mathbb{R}^n$ that maps $\mathbb{Z}^n$ to $\mathbb{Z}^n$. Such an A gives rise to map $a : T^n \to T^n$. Show that we can identify $H_1(T^n)$ with Z^n in such a way that the map $a_* : H_1(T^n) \to H_1(T^n)$ coincides with $A : \mathbb{Z}^n \to \mathbb{Z}^n$.

Problem 5. Let K be a simplicial complex in $\mathbb{R}^n$ that is the union of two subcomplexes K_1, K_2. Check that $K_1 \cap K_2$ is also a simplicial complex. Give an elementary proof for the Mayer-Vietoris sequence in simplicial homology. (The same proof must also work for simplicial homology of Δ-complexes.)