MAT 539 Algebraic Topology

Problem Set 1

due Wednesday, September 22

Please do the following questions from Hatcher, as well as two additional questions (below).

5, 8 p.131.
(Don't worry too much about Δ-complexes; homology works just as well as for simplicial complexes we defined, except you have more freedom gluing up simplices. Read up the precise definitions in Hatcher if you must.)

11 p.132. Use this and homology of spheres to show that the n-ball D^n doesn't retract onto its boundary sphere S^{n-1}.

16 p.132.

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The following two questions, especially the second, deal with homology with coefficients in abelian groups other than $\mathbb{Z}$. The construction is very similar: the group of singular n-chains with coefficients in an abelian group G, denoted $C_n(X;G)$, is defined as

$$C_n(X;G) = \{\text{finite sums of the form } \sum m_i\sigma_i\},$$

where $\sigma_i : \Delta^n \to X$ are singular n-simplices, and $m_i \in G$. The boundary maps are defined as before. The simplest case (in fact simpler than integer coefficients) is coefficients in $\mathbb{Z}/2$, where you don't have to worry about signs, and the chain groups and homology groups are vector spaces over $\mathbb{Z}/2$. In general, homology groups with coefficients in G are G-modules.

Problem 1. (i) Define a singular n-cube in a space X to be a continuous map $\sigma : I^n \to X$. Using $\mathbb{Z}/2$ coefficients for simplicity, define the cubical chain group of X, $C_n^{\square}(X;\mathbb{Z}/2)$, to be the $\mathbb{Z}/2$ vector space with generators the singular n-cubes. Using geometric intuition as in the simplicial case, define a boundary map ∂ to make these groups into a chain complex. Verify that $\partial \circ \partial = 0$. Denote by $H_n^{\square}(X;\mathbb{Z}/2)$ the resulting "homology groups" of X.

(ii) Show that we do not obtain the ordinary singular homology groups of $H_n(X;\mathbb{Z}/2)$ by this recipe.

(iii) Compute $H_1^{\square}(S^1;\mathbb{Z}/2)$.

(A similar construction can be found in a book by Massey, *A basic course in algebraic topology*, GTM 127. Massey does singular homology using cubes, but modifies the above definition so as to recover the expected singular homology groups.)

Problem 2. Given a short exact exact sequence of abelian groups

$$0 \longrightarrow G \longrightarrow G' \longrightarrow G'' \longrightarrow 0,$$

and a space X, one has a short exact sequence

$$0 \longrightarrow C_n(X;G) \longrightarrow C_n(X;G') \longrightarrow C_n(X;G'') \longrightarrow 0,$$

and hence a long exact sequence in homology. (Here, as usual, $C_n(X;G)$ stands for the group of n-chains of X with coefficients in G.) The homomorphism

$$\beta_* : H_n(X;G'') \to H_{n-1}(X;G),$$

arising in the long exact sequence, is called the Bockstein homomorphism associated with the given coefficient sequence. The same construction works for simplicial homology of a Δ-complex as well.

Compute β_* for the coefficient sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

and $\mathbb{R}P^2$ with the Δ-complex structure from p. 102 of Hatcher.