MAT 530, Homework 6, due Monday, 10/5

For finite products of topological spaces, there is essentially only one way to define a reasonable product topology. For infinite products, the "correct" definition is not so obvious. The purpose of this homework is to discuss various topologies on (infinite) products.
Let $\{X_\alpha\}$ be a collection of topological spaces, indexed by $\alpha \in A$, where the index set A is infinite (and not necessarily countable). Let $X = \prod_{\alpha \in A} X_a$ be the product of sets X_α; points of X can be given by their coordinates, $\mathbf{x} = (x_\alpha)_{\alpha \in A}$.

1. (a) Consider the collection $\mathcal{B}$ of all subsets of X of the form $\prod_{\alpha \in A} U_\alpha$, where U_α is open in X_α. Check that $\mathcal{B}$ gives a basis for some topology on X. (This topology is called the *box topology.*)

(b) Consider the collection $\mathcal{B}$ of all subsets of X of the form $\prod_{\alpha \in A} U_\alpha$, where U_α is open in X_α, and $U_\alpha = X_\alpha$ except for finitely many values of α. Check that $\mathcal{B}$ gives a basis for some topology on X. (This topology is called the *product topology.*) The product topology is the most important topology on the product space; for most purposes this is the "correct" choice of topology.

(c) For the product of countably many copies of $\mathbb{R}$,

$$\mathbb{R}^\infty = \{(x_1, x_2, x_3, ...)\},$$

consider two topologies as above, the box topology and the product topology. Let $f : \mathbb{R} \to \mathbb{R}^\infty$ be given by

$$f(t) = (t, t, t, t, \dots).$$

Show that f is continuous in the product topology but not in the box topology.

(d) More generally, consider $X = \prod_{\alpha \in A} X_a$, equipped with the product topology, and a function $f : S \to X$ for some topological space S. Let $f_\alpha : S \to X_\alpha$ be coordinate functions for f, so that

$$f(s) = (f_\alpha(s))_{\alpha \in A}.$$

Prove that f is continuous if and only if every f_α is continuous.

2. (a) Consider $\mathbb{R}^\infty$ as above, with three different topologies: the box topology, the product topology, and the metric topology given by the *sup*-metric on sequences. Show that these topologies are indeed all different; compare them if possible (that is, determine which one is finer/coarser), and justify your answers.

(b) Find (with proof) a metric that induces the product topology on $\mathbb{R}^\infty$.

(c) Show that the box topology is not metrizable; indeed, it is not even first countable. You can show this directly or use part (d).

(d) Consider $C \subset \mathbb{R}^\infty$ given by points with positive coordinates,

$$C = \{(x_1, x_2, x_3, \dots) |\ x_i > 0 \text{ for every } i\}.$$

Show that the point $\mathbf{0} = (0, 0, 0, \dots)$ is in the closure of C with respect to the box topology, however, no sequence of points of C converges to $\mathbf{0}$.

3. The Tychonoff theorem says that the product of compact spaces is always compact (when equipped with product topology). In the infinite case, proving this fact requires the axiom of choice and is much harder than in finite case. (See the Munkres book if you are interested.) Note that the product of infinitely many compact spaces *will not* typically be compact if equipped with the box topology or the *sup*-metric topology.

Prove "Tychonoff lite": the product of countably many compact metric spaces is compact (in the product topology).