MAT 530, Homework 13, due Wednesday, 12/2

Questions 1-8 provide a review of the algebraic topology part of the course for the final exam and the comps. Questions 9-10 are not mandatory; I recommend that you think about them, but you don’t have to submit solutions. (Complete solutions of these two might be a bit tedious to write down.)


2. (a) Find a group $G$ acting on the torus $T$ (describe the action) so that $T/G$ is homeomorphic to $T$.
(b) Find a group $G$ acting on the torus $T$ (describe the action) so that $T/G$ is homeomorphic to the Klein bottle $K$. (Thus, you get a covering $T \rightarrow K$.)
(c) Prove that there is no covering $K \rightarrow T$.

3. (a) Show that any 2-fold covering has a non-trivial deck transformation.
(b) Give an example of a 3-fold covering without any non-trivial deck transformations.
( Assume that all spaces are “nice” - path-connected, locally path-connected.)

4. Let $S$ be the solid torus $S = D^2 \times S^1$, and $K$, $L$ two simple closed curves in $S$, as shown in the picture. (Everything is a subset of $\mathbb{R}^3$.) Notice that if you are allowed to move $K$ and $L$ outside the solid torus (over the hole), then these two curves can be untangled from one another (visual intuition suggests that $K$ and $L$ cannot be untangled inside $S$).
Prove that $K$ is not null-homotopic in the complement of $L$ in $S$. (Think of $K$ as the image of a loop $\gamma : S^1 \rightarrow S \setminus L$; you need to show that there is no homotopy $F : S^1 \times I \rightarrow S \setminus L$ such that $F(t, 0) = \gamma(t)$ and $F(t, 1) = \text{const.}$)

5. Let $X$ be a path-connected, locally path-connected space with a finite fundamental group. Show that any map $f : X \rightarrow S^1$ is null-homotopic.

6. Let $T$ be a torus and $C$ a homotopically non-trivial, simple closed curve in $T$. Let $S$ be a torus with an open disk removed. Let $X$ be a space obtained by gluing $S$ to $T$ so that the boundary $\partial S$ of the surface $S$ is identified with $C$.

(a) Write a presentation for $\pi_1(X)$. 
(b) Show that \( C \) must lift to a closed loop (as opposed to an open path) on any 2-fold cover of \( X \).

7. Let \( S^n \) be the unit sphere in \( \mathbb{R}^{n+1} \) and let \( s_0 \) be the point \((1,0,\ldots,0) \in S^n \). Define \( \pi_n(X, x_0) \) to be the set of homotopy classes of maps \( f : (S^n, s_0) \to (X, x_0) \). (We assume that homotopies fix the basepoint.)

Let \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) be a covering. Define the map \( p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0) \) by \( p_*([f]) = [p \circ f] \).

(a) Show that \( p_* \) is well-defined.
(b) Show that for \( n \geq 2 \), \( p_* \) is a bijection.

8. Think of \( S^3 \) as \( \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \). Fix two relatively prime positive integers \( n, k \). Consider the action of the cyclic group \( \mathbb{Z}_n \) of order \( n \) on \( S^3 \), such that the generator \( h \) of \( \mathbb{Z}_n \) acts by

\[
h(z_1, z_2) = (z_1 e^{2\pi i/n}, z_2 e^{2\pi ik/n}).
\]

(Check that \( h^n \) is the identity map.) The quotient space of this action, \( S^3/\mathbb{Z}_n \), is called the lens space \( L(n, k) \).

Show that if \( L(n, k) \) and \( L(n', k') \) are homeomorphic, then \( n = n' \).

Optional questions, do not submit:
9. Use topology to show that any subgroup of a free group is free. (To avoid technicalities but still see the idea, you can make extra assumptions: the group is finitely generated, the subgroup has a finite index, etc.)

10. Show that any finite cellular space embeds into a Euclidean space of sufficiently high dimension. (Use 42.N.2 in Viro’s book.)