MAT 530, Homework 12, due Wednesday, 11/18

You need to work through these questions by Wednesday, Nov 18, because the questions reinfornce the material that will be on the exam. However, written solutions will be accepted until Friday, Nov 20.

1. Let $p : \tilde{X} \to X$ be a covering, X a path-connected space. Consider f, a loop in X based at x_0; we can lift f to paths in $\tilde{X}$ starting at different points of the fiber $p^{-1}(x_0)$. It turns out that we may get one lift which is a closed loop, and another which is not.

(a) Let X be a figure 8 space. Give an example of a 3-fold covering $p : \tilde{X} \to X$, a loop f based at $x_0 \in X$, and two points $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$, such that a lift of f staring at $\tilde{x}_1$ is a closed loop, and a lift starting at $\tilde{x}_2$ is not.

(b) show that the following two conditions are equivalent:
(i) For every loop f based at $x_0 \in X$, either all lifts of f to $\tilde{X}$ are closed loops, or none of the lifts are closed (regardless of the starting points of the lifts in the fiber over x_0).
(ii) $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is a normal subgroup of $\pi_1(X, x_0)$ for some $\tilde{x}_0 \in p^{-1}(x_0)$.
Hint: what is the relation between the subgroups $p_*(\pi_1(\tilde{X}, \tilde{x_1}))$ and $p_*(\pi_1(\tilde{X}, \tilde{x_2}))$ for different $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$?

(c) If conditions of (b) are satisfied, $p : \tilde{X} \to X$ is called a *normal* covering. Check that this notion is independent of the choice of $x_0 \in X$.

(d) Show that every two-fold covering is normal.

2. Prove that any continuous map $f : S^2 \to T^2$ is null-homotopic. **Hint:** use the covering $R^2 \to T^2$.

3. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ to be an octagon the the plane (with the standard topology), and consider the space obtained by gluing together all of the sides of the octagon in a way that preserves the cyclic order of vertices (so that A_1A_2 is identified with A_iA_{i+1} via a linear homeomorphism sending A_1 to A_i and A_2 to A_{i+1}; A_8A_1 is glued to A_1A_2 so that $A_8 \in A_8A_1$ is identified with $A_1 \in A_1A_2$, $A_1 \in A_8A_1$ is identified with $A_2 \in A_1A_2$). The resulting space X has the quotient topology.

(a) Compute $\pi_1(X)$.

(b) Describe all the covering spaces of X. Explain how the classification and hierarchy theorems work in this case.

(c) Is X a surface? **Prove** your answer.

Please also do questions 14, 15 on p. 80 in Hatcher.

Please also do questions 43.Px, 43.Qx, 43.Rx, 43.Ux about path-connectness and connectedness of cellular spaces in Viro's book. (These questions are all related; please hand in everything.)

Read the discussion of the covering spaces for the figure 8 space in Hacther. Note the following corollary: the free group on 2 generators has a subgroup isomorphic to the free group on k generators, for *any* $k \geq 1$.