1. Let \( p : \tilde{X} \to X \) be a covering, \( X \) a path-connected space. Consider \( f \), a loop in \( X \) based at \( x_0 \); we can lift \( f \) to paths in \( \tilde{X} \) starting at different points of the fiber \( p^{-1}(x_0) \). It turns out that we may get one lift which is a closed loop, and another which is not.

(a) Let \( X \) be a figure 8 space. Give an example of a 3-fold covering \( p : \tilde{X} \to X \), a loop \( f \) based at \( x_0 \in X \), and two points \( \tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0) \), such that a lift of \( f \) starting at \( \tilde{x}_1 \) is a closed loop, and a lift starting at \( \tilde{x}_2 \) is not.

(b) Show that the following two conditions are equivalent:

(i) For every loop \( f \) based at \( x_0 \in X \), either all lifts of \( f \) to \( \tilde{X} \) are closed loops, or none of the lifts are closed (regardless of the starting points of the lifts in the fiber over \( x_0 \)).

(ii) \( p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \) is a normal subgroup of \( \pi_1(X, x_0) \) for some \( \tilde{x}_0 \in p^{-1}(x_0) \).

**Hint:** what is the relation between the subgroups \( p_*(\pi_1(\tilde{X}, \tilde{x}_1)) \) and \( p_*(\pi_1(\tilde{X}, \tilde{x}_2)) \) for different \( \tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0) \)?

(c) If conditions of (b) are satisfied, \( p : \tilde{X} \to X \) is called a normal covering. Check that this notion is independent of the choice of \( x_0 \in X \).

(d) Show that every two-fold covering is normal.

2. Prove that any continuous map \( f : S^2 \to T^2 \) is null-homotopic. **Hint:** use the covering \( R^2 \to T^2 \).

3. Let \( A_1A_2A_3A_4A_5A_6A_7A_8 \) to be an octagon the the plane (with the standard topology), and consider the space obtained by gluing together all of the sides of the octagon in a way that preserves the cyclic order of vertices (so that \( A_1A_2 \) is identified with \( A_iA_{i+1} \) via a linear homeomorphism sending \( A_1 \) to \( A_i \) and \( A_2 \) to \( A_{i+1} \); \( A_8A_1 \) is glued to \( A_1A_2 \) so that \( A_8 \in A_8A_1 \) is identified with \( A_1 \in A_1A_2 \), \( A_1 \in A_8A_1 \) is identified with \( A_2 \in A_1A_2 \)). The resulting space \( X \) has the quotient topology.

(a) Compute \( \pi_1(X) \).

(b) Describe all the covering spaces of \( X \). Explain how the classification and hierarchy theorems work in this case.

(c) Is \( X \) a surface? **Prove** your answer.

Please also do questions 14, 15 on p. 80 in Hatcher.

Please also do questions 43.Px, 43.Qx, 43.Rx, 43.Ux about path-connectness and connectedness of cellular spaces in Viro’s book. (These questions are all related; please hand in everything.)

Read the discussion of the covering spaces for the figure 8 space in Hatcher. Note the following corollary: the free group on 2 generators has a subgroup isomorphic to the free group on \( k \) generators, for any \( k \geq 1 \).