MAT 530 Topology, Geometry I

Problem Set 5

due Friday, October 9

Problem 1. Consider the product space $\{0,1\}^{[0,1]}$, where $\{0,1\}$ has discrete topology. As a set, this space can be identified with the set of all functions on $[0,1]$ that take only two values, 0 and 1. The Tychonoff theorem (to be proved next week) implies that $\{0,1\}^{[0,1]}$ is compact (with the product topology). Show that, however, it is not sequentially compact, by constructing a sequence with no convergent subsequences.

Problem 2. (Still hoping to prove the Tychonoff soon...) In the meanwhile, prove that a countable product of sequentially compact spaces is sequentially compact. In particular, this gives a simple Tychonoff-free proof that a countable product of compact metric spaces is compact.

Problem 3. Recall that the Cantor set is constructed by starting with the interval $[0,1]$, removing the middle third $I_1^{(1)} = (1/3, 2/3)$, then removing the middle thirds $I_2^{(1)} = (1/9, 2/9)$ and $I_2^{(2)} = (7/9, 8/9)$, then removing the middle thirds $I_3^{(i)}$ from the four remaining closed intervals, and so on. The Cantor set is then defined as

$$K = [0,1] - \bigcup_{i,j} I_j^{(i)}.$$

The elements of the Cantor set can be written as infinite sequences of 0's and 1's: let the first digit of the sequence for $x \in K$ be 0 if $x \in [0,1/3]$, 1 if $x \in [2/3,1]$; similarly, the second digit is 0 if x is in the left third of the corresponding interval (after the removal of $I_2^{(1)}$ and $I_2^{(2)}$), and 1 if it is in the right third, and so on.
Consider K as a subspace of $\mathbb{R}$ (with the standard topology). Show that K is homeomorphic to $\{0,1\}^\omega$, where the latter space has the product topology, and the two-point space $\{0,1\}$ has the discrete topology.

Problem 4. There exist non-metrizable topological spaces where each point has a neighborhood homeomorphic to $\mathbb{R}$. One such example is given by the *long line*, to be discussed in Monday lecture. To give another (much simpler) example, construct a non-Hausdorff topological space where each point has a neighborhood homeomorphic to $\mathbb{R}$.

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Please also do questions 7 of §19, 6 of §20, 2 of §25 from Munkres.