MAT 530 Topology, Geometry I

Problem Set 4

due Friday, October 2

Problem 1. Suppose that the topological space X has a countable basis. Show that X is compact if and only if every sequence in X has a convergent subsequence.
Remark. In general, neither of implications above has to be true. We will see some examples later.

Problem 2. Let $[0,1]^\omega = [0,1] \times [0,1] \times [0,1] \times \dots$ the product of countably many copies of $[0,1]$. Equip this set with the topology $\mathcal{T}$ whose basis is given by sets of the form $U_1 \times U_2 \times U_3 \times \dots$, where each U_i is an open subset of $[0,1]$, and $U_i = [0,1]$ **for all but finitely many** $i = 1, 2, 3, \dots$. Please convince yourself that this is indeed a basis, but do not hand the proof. The resulting topology is the *product topology* on the infinite product.

Show that $([0,1]^\omega, \mathcal{T})$ is compact.

(Yes, compactness in the product topology follows from the Tychonoff theorem, but this is an elementary special case. Please give a direct proof; do not refer to the Tychonoff theorem.)

Problem 3. Suppose X is compact, Hausdorff, and

$$A_1 \supset A_2 \supset A_3 \supset \dots$$

are closed connected subsets. Show that the intersection

$$\cap_{n=1}^{\infty} A_n$$

is connected.

Suppose the requirement that X be compact is dropped. Show that $\cap A_n$ can be disconnected. (Remark: the empty set is connected.)

Problem 4. Let D be the closed disk of radius 1. Consider the set P_n of convex polygons P such that:

- P is contained in D.
- The perimeter of P is at most 1.
- P has at most n vertices. (The cases $n = 1, 2$ are included; the perimeter of a 2-gon is twice the distance between vertices.)

Define a natural metric on P_n.

Show that P_n is compact.

Show that there's a polygon in P_n with the maximal area.

Show that this polygon is the regular n-gon.

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Please also do questions 5 of §26 and 2 of §27 from Munkres. For question 2 of §27, read the definition on p. 175.