MAT 530 Topology, Geometry I

Problem Set 2

due Friday, September 18

Problem 1. Let $\mathbb{N}$ be the set of positive integers, and consider all infinite arithmetic progressions of the form $\{am+b\}_{m=0,1,2,\dots}$, where a and b positive integers. Prove that they form a basis for some topology on $\mathbb{N}$.

Show that if the set of prime numbers were finite, then $\{1\}$ would be open in this topology.

Using the above, prove that there are infinitely many prime numbers.

(Please give an honest new "topological" proof; rewriting the usual Euclid's proof in topological terms will be worth little credit.)

Problem 2. Let X be a topological space, $A \subset X$. Suppose that there exists a sequence $\{x_n\}$ of points of A converging to a. Show that $a \in \bar{A}$.

Show that the converse is true in metric spaces, i.e if (X, d) is a metric space, $A \subset X$, and $a \in \bar{A}$, then there exists a sequence $\{x_n\}$ of points $x_n \in A$ that converges to a. The next problem demonstrates that the result may *fail* in a general topological space.

Problem 3. Consider the set

$$X = \{(x_1, x_2, x_3, \dots) : x_i \in \mathbb{R}\}$$

of all sequences of real numbers. (In other words, $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$.)

Show that all sets of the form $U_1 \times U_2 \times U_3 \dots$, where U_i is open in the standard topology on $\mathbb{R}$, form a basis of a topology on X.

In the topological space X (with the topology generated by the above basis), consider

$$A = \{(x_1, x_2, x_3 \dots) : x_i > 0 \text{ for all } i > 0\}.$$

Show that the point $\mathbf{0} = (0, 0, 0, \dots)$ belongs to the closure of A, however, no sequence of points of A converges to $\mathbf{0}$.

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Please also do questions 8 of §16; 6, 7, and 19 of §17 in Munkres.

Required Reading: please read §18.