Problem 1. Show that every open set in the standard topology on $\mathbb{R}$ is a countable disjoint union of open intervals.

Problem 2. Metrics $d_1$ and $d_2$ on a set $X$ are called equivalent if there are constants $C, c > 0$ such that $cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y)$ for all $x, y \in X$. Show that equivalent metrics induce the same topology on $X$.

Problem 3. Let $C$ be the space of real-valued continuous functions on $[0, 1]$. For $f, g \in C$ define

$$d_1(f, g) = \int_0^1 |f(x) - g(x)|\,dx, \quad d_{sup}(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|.$$ 

Show that $d_1$ and $d_{sup}$ are metrics on $C$. Prove that the topologies induced on $C$ by these metrics are different. Is it true that one of them is finer than the other?

Problem 4. Let $(X, d)$ be a metric space. Consider $Y = X \cup \{a\}$, where $a \not\in X$ (i.e. $Y$ is $X$ with an extra point added). Define collections $T_1, T_2$ of subsets of $Y$ as follows:

- $U \in T_1$ iff either $U \subset X$ and $U$ is open in $(X, d)$, or $U = Y$.
- $U \in T_2$ iff either $U = \emptyset$, or $U = V \cup \{a\}$, where $V \subset X$ is open in $(X, d)$.

Check whether each of $T_1, T_2$ is a topology on $Y$, and if so, whether it can be induced by any metric on $Y$.

Problem 5. Let $B$ be a basis for the standard topology on $\mathbb{R}$. Prove that $B$ can always be decreased, i.e there is a set $U \in B$ such that $B - \{U\}$ is still a basis for the standard topology.

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Please also do questions 1, 5 (basis part only) and 8b of Munkres §13.