MAT 364 - Homework 5 Solutions

Exercise 3.7: Let X be a topological space an A a subset of X. Prove that $B \subset A$ is closed in A if and only if $B = A \cap C$ for some $C \subset X$ closed in X.

Solution: ($\Rightarrow$) Let $B \subset A$ be closed in A. Then by definition the complement $A - B$ is an open set in A. From the statement about open sets in Theorem 3.12, we have that $A - B = A \cap U$ for some open set $U \subset X$. Then

$$B = A - (A - B) = A - (A \cap U) = (X \cap A) - (A \cap U) = (X - U) \cap A$$

Since U is open in X, $X - U$ is closed in X, so if we consider the closed set $C = X - U$, we have $B = C \cap A$ as desired.

($\Leftarrow$) Let $B = A \cap C$ for some $C \subset X$ closed in X. Then

$$A - B = A - (A \cap C) = (X - C) \cap A$$

and $X - C$ is open in X since C is closed. Thus the "openness" section of Theorem 3.12 gives that $A - B$ is an open set in A, so that B is closed in A.

Exercise 3.28: Let X, Y be topological spaces, and let $X \times Y$ be given the product topology. Prove that the projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ defined by $p_X(x, y) = x$ and $p_Y(x, y) = y$ are continuous.

Solution: We'll just show that p_X is continuous - basically the same argument shows that p_Y is continuous.

Let $U \subset X$ be an open set. Then the preimage under the projection is

$$(p_X^{-1})(U) = \{(x, y) \in X \times Y : p_X(x, y) \in U\} = \{(x, y) \in X \times Y : x \in U\} = U \times Y$$

The product topology on $X \times Y$ is generated by the basis of sets of the form $O \times O'$, where O, O' are open sets in X and Y, respectively. We observe that $U \times Y$ is of this form - $U \subset X$ is open by assumption, and $Y \subset Y$ is open by the definition of a topology. Therefore we have that $U \times Y$ is a basis element of the product topology on $X \times Y$, and is in particular an open set in the product topology.

Thus for any $U \subset X$ open, the preimage $(p_X^{-1})(U)$ is open in $X \times Y$, and therefore p_X is continuous.

Question 1: Let $\mathcal{B}, \mathcal{B}'$ be two bases for X, equivalent in that they satisfy the conditions

(1) For every $B \subset \mathcal{B}$ and every $x \in B$, there exists a $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.
(2) For every $B' \subset \mathcal{B}'$ and every $x \in B'$, there exists a $B \in \mathcal{B}$ such that $x \in B \subset B$.

Show that $\mathcal{B}$ and $\mathcal{B}'$ generate the same topology on X.

Solution: Let $\mathcal{T}, \mathcal{T}'$ be the topologies generated by $\mathcal{B}, \mathcal{B}'$, respectively. We wish to show that $\mathcal{T} = \mathcal{T}'$. We begin by showing that $\mathcal{T} \subset \mathcal{T}'$.

Let $U \in \mathcal{T}$ be given. Then we wish to show that $U \in \mathcal{T}'$. To show this, it suffices to show that $\mathcal{B} \subset \mathcal{T}'$, that is, that every basis element of $\mathcal{B}$ is in fact an open set in the $\mathcal{T}'$ topology. This is because if $U \in \mathcal{T}$, then $U = \cup_\alpha B_\alpha$ for some collection $\{B_\alpha\}$ of elements of $\mathcal{B}$. If each of these B_α is in fact open in $\mathcal{T}'$, then U is an arbitrary union of open sets of $\mathcal{T}'$, hence is an open set in $\mathcal{T}'$.

Thus we need to show that $\mathcal{B} \subset \mathcal{T}'$. Let $B \in \mathcal{B}$ be given. Let $x \in B$ be given as well. Then from property (1), there exists $B'_x \in \mathcal{B}$ such that $x \in B'_x \subset B$. Repeating this for every $x \in B$, we observe that $B = \cup_{x\in B} B'_x$. Each B'_x is an open set in the $\mathcal{T}'$ topology, so the union $\cup_{x\in B} B'_x$ is an open set in $\mathcal{T}'$, that is, B is open in the $\mathcal{T}'$ topology. We therefore have that $\mathcal{B} \subset \mathcal{T}'$, which as discussed in the previous paragraph shows that $\mathcal{T} \subset \mathcal{T}'$.

The same argument, suitably changed, shows that $\mathcal{T}' \subset \mathcal{T}$. You show as above that the result follows if $\mathcal{B}' \subset \mathcal{T}$, and use property (2) above to show that $\mathcal{B}' \subset \mathcal{T}$. Then the two inclusions $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T}' \subset \mathcal{T}$ give the equality $\mathcal{T} = \mathcal{T}'$.

Question 2: Let $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$ be the collection of open intervals in $\mathbb{R}$ with rational endpoints. Show that

(1) $\mathcal{B}$ is a basis for *some* topology on $\mathbb{R}$.
(2) The topology generated by $\mathcal{B}$ is the usual Euclidean topology on $\mathbb{R}$.

Solution: For the first part, we need to check the two conditions in the definition of a basis, items (1) and (2) from Definition 3.3 in the text.

The first condition is that $\mathcal{B}$ should cover $\mathbb{R}$, that is, that $\cup_{B\in\mathcal{B}} B = \mathbb{R}$. Consider the intervals $(j, j+2)$ for $j \in \mathbb{Z}$, each of which is an element of $\mathcal{B}$ since $\mathbb{Z} \subset \mathbb{Q}$. The the union $\cup_{j\in\mathbb{Z}}(j, j+2) = \mathbb{R}$, and since $\cup_{j\in\mathbb{Z}}(j, j+2) \subset \cup_{B\in\mathcal{B}} B$ we have $\mathbb{R} \subset \cup_{B\in\mathcal{B}} B$. The other inclusion is obvious, so $\cup_{B\in\mathcal{B}} B = \mathbb{R}$.

The second condition is that, for every $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$, there must exist $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. We first consider what this intersection looks like. Say $B_1 = (a, b)$ and $B_2 = (c, d)$ for $a, b, c, d \in \mathbb{Q}$. Then if $B_1 \cap B_2 \neq \emptyset$, we have three possibilities - either $B_1 \subset B_2$, $B_2 \subset B_1$, or the intervals B_1 and B_2 overlap but neither is contained in the other. In the first two cases, the intersection $B_1 \cap B_2$ is B_1 or B_2, respectively, and in the third case, the intersection is the interval (c, b), which is an interval with rational endpoints. Thus in all cases we have that $B_1 \cap B_2$ is itself an element of the basis $\mathcal{B}$, so setting $B_3 = B_1 \cap B_2$ gives the desired neighborhood.

We now check that the topology generated by $\mathcal{B}$ is the same as the Euclidean topology on $\mathbb{R}$. Recall that the standard Euclidean topology on $\mathbb{R}$ is generated by the basis $\mathcal{B}' = \{D(x, r) : x \in \mathbb{R}, r > 0\}$ of "open balls", which in $\mathbb{R}$ are just open intervals of the form $(x - r, x + r)$. Instead of writing our intervals in this form, we can just write them as (a, b) for any $a, b \in \mathbb{R}$, and observe that any open interval of the form (a, b) can be written in the form $(x - r, x + r)$ for appropriate choice of x, r. Thus the standard Euclidean topology on $\mathbb{R}$ is generated by the basis $\mathcal{B}' = \{(a, b), a < b \in \mathbb{R}\}$.

To show that the topology generated by $\mathcal{B}$ is the standard Euclidean topology, we'll show that the bases $\mathcal{B}$ and $\mathcal{B}'$ are equivalent, so that from Question 1 above they generate the same topology. We need to show both conditions (1) and (2) stated in that question.

To show condition (1), let $B \in \mathcal{B}$ be given, and let $x \in B$ also be given. Note that $B = (a, b)$ for some $a, b \in \mathbb{Q} \subset \mathbb{R}$, so in fact we can consider B as an element of the $\mathcal{B}'$ basis. (An open interval with rational endpoints is automatically an open interval with real endpoints). Setting $B' = B$, we have that $x \in B' \subset B$ as desired.

Condition (2) is slightly more complicated. Let $(a, b) = B' \in \mathcal{B}'$ be given, so that $a, b \in \mathbb{R}$, and let $x \in (a, b)$. We'll use the following fact about the real numbers - between any two real numbers, there is a rational number. Using this property twice, we can find rational numbers c, d such that $a < c < x < d < b$. Now $(c, d) \in \mathcal{B}$ since it is an open interval with rational endpoints. Therefore we have that $x \in (c,d) = B \subset (a,b) = B'$, and condition (2) is fulfilled.

Question 3: Let $\mathcal{B} = \{[a,b], a,b \in \mathbb{R}\}$ be the collection of all closed intervals in $\mathbb{R}$. Can $\mathcal{B}$ be a basis of *some* (not necessarily standard) topology on $\mathbb{R}$? Why or why not?

Solution: We need to check to see if the collection $\mathcal{B}$ satisfies the two requirements in the definition of a basis, items (1) and (2) from Definition 3.3 in the text. It is not hard to see that condition (1) is satisfied, but condition (2) will not be true. Therefore $\mathcal{B}$ cannot be the basis for a topology.

To see that condition (2) fails, consider $B_1 = [a,b]$ and $B_2 = [b,c]$. The intersection is $B_1 \cap B_2 = \{b\}$. In order for condition (2) to be satisfied, we need to find a third basis element $B_3 = [d,e]$ such that $b \in [d,e] \subset \{b\} = B_1 \cap B_2$. This is not possible, as any closed interval containing b will contain points other than b, hence will not be a subset of $B_1 \cap B_2$.

Question 4: Consider $X = \{a,b,c,d,e\}$, with a collection of subsets $\mathcal{T} = \{\{a,b\},\{d,e\}, \{a,b,d,e\}, X, \emptyset\}$. Prove that $\mathcal{T}$ is a topology on X. Find two subsets of X, A and B, each containing more than one point each, such that the subspace topology on A is discrete and the subspace topology on B is indiscrete. Explain your answer.

Solution: We begin by checking that $\mathcal{T}$ satisfies the 3 conditions of a topology - that $X, \emptyset \in \mathcal{T}$, that $\mathcal{T}$ is closed under arbitrary union, and that $\mathcal{T}$ is closed under finite intersection.

That $X, \emptyset \in \mathcal{T}$ is clear. To check that $\mathcal{T}$ is closed under union, we have that any element of $\mathcal{T}$ union either itself, X, or $\emptyset$ is evidently in T. The remaining unions are

$$\{a,b\} \cup \{a,b,d,e\} = \{a,b,d,e\} \qquad \{d,e\} \cup \{a,b,d,e\} = \{a,b,d,e\} \qquad \{a,b\} \cup \{d,e\} = \{a,b,d,e\}$$

so we have that $\mathcal{T}$ is closed under union. (Note that the last union here is why the original statement of the problem, which didn't include $\{a,b,d,e\}$ in $\mathcal{T}$, was incorrect.) By induction $\mathcal{T}$ is closed under finite unions, and this is enough to show that $\mathcal{T}$ is closed under arbitrary unions since $\mathcal{T}$ is finite.

To check that $\mathcal{T}$ is closed under intersection, we compute all the intersections. We'll prove that the intersection of any two elements of $\mathcal{T}$ is again in $\mathcal{T}$, and induction then allows us to say that $\mathcal{T}$ is closed under finite intersection. Any subset intersected with itself, X, or $\emptyset$ is again easily seen to be in $\mathcal{T}$. Checking the remaining possible intersections, we have

$$\{a,b\} \cap \{a,b,d,e\} = \{a,b\} \qquad \{d,e\} \cap \{d,e\} = \{a,b,d,e\} \qquad \{a,b\} \cap \{d,e\} = \emptyset$$

so we have that $\mathcal{T}$ is closed under intersection.

Knowing that $\mathcal{T}$ is a topology, we move on the the second part. Multiple answers are possible for this part; one possible answer is to choose $A = \{a,d\}$ and $B = \{a,b\}$.

We consider A first. A has the subspace topology, so the open sets in A are obtained by taking open sets in $\mathcal{T}$ and intersecting with A. They are

$$\begin{aligned}
\{a,b\} \cap A &= \{a\} \\
\{d,e\} \cap A &= \{d\} \\
\{a,b,d,e\} \cap A &= A \\
X \cap A &= A \\
\emptyset \cap A &= \emptyset,
\end{aligned}$$

or $\mathcal{T}_A = \{\{a\}, \{d\}, A, \emptyset\}$. These are all of the subsets of A, therefore A has the discrete topology.

To find the topology on B, we repeat this procedure,

$$\begin{aligned}
\{a, b\} \cap B &= B \\
\{d, e\} \cap B &= \emptyset \\
\{a, b, d, e\} \cap B &= B \\
X \cap B &= B \\
\emptyset \cap B &= \emptyset,
\end{aligned}$$

or $\mathcal{T}_B = \{B, \emptyset\}$. Since the only open sets for B are itself and $\emptyset$, B has the indiscrete topology.