CONNECTED SPACES AND HOW TO USE THEM

1. How to prove X is connected

Checking that a space X is NOT connected is typically easy: you just have to find two disjoint, non-empty subsets A and B in X, such that $A \cup B = X$, and A and B are both open in X. (Notice that when X is a subset in a bigger space, say $\mathbb{R}^n$, "open in X" means relatively open with respected to X. Be sure to check carefully all required properties of A and B.)

To prove that X is connected, you must show no such A and B can ever be found - and just showing that a particular decomposition doesn't work is not enough. Sometimes (but very rarely) open sets can be analyzed directly, for example when X is finite and the topology is given by an explicit list of open sets. (Anoth example is indiscrete topology on X: since the only open sets are X and $\emptyset$, no decomposition of X into the union of two disjoint open sets can exist.) Typically, however, there are too many open sets to argue directly, so we develop several tools to establish connectedness.

Theorem 1. *Any closed interval $[a, b]$ is connected.*

Proof. Several proofs are possible; one was given in class, another is Theorem 2.28 in the book. ALL proofs are a mix of analysis and topology and use a fundamental property of real numbers (Completeness Axiom). □

Corollary 1. *All open and half-open intervals are connected; all rays are connected; a line is connected.*

Proof. This follows from the theorem (argue by contradiction). For example, suppose that the ray $[0, +\infty)$ is disconnected, ie $[0, +\infty) = A \cup B$, where A and B are non-empty, disjoint, and relatively open. Pick a point $a \in A$, $b \in B$, and consider the closed interval $I = [a, b]$ (we can assume $a < b$). Then the sets $A \cap I$ and $B \cap I$ are non-empty, disjoint, and relatively open in I (check!). This means that I is not connected, a contradiction. □

Theorem 2. *A union of two intersecting connected (sub)spaces is connected. NAmely, suppose $X = U \cup V$, where U, V are both connected, and $U \cap V \neq \emptyset$. Then X is connected.*

Proof. Suppose X is not connected, $X = A \cup B$, where A and B are non-empty, disjoint, and open. Pick a point $v \in U \cap V$. We can assume $v \in A$. Now, consider sets $U \cap A$ and $U \cap B$. They are disjoint and relatively open in U (why?); $U \cap A$ contains v and so is non-empty. If $U \cap B$ is non-empty as well, we would get that U is disconnected; since it is given that U is connected, $U \cap B$ must be empty, and so $U \subset A$. The same reasoning shows that $V \subset B$. But then $X = U \cup V$ is contained in A, and B must be empty. Comtradiction. □

Remark 1. The same proof goes through to show that a union of arbitrary number of connected spaces sharing a point is connected. See also Lemma 3.30 in the textbook.)

Path-connectedness is a property that is longer to define but easier to establish than connectedness. Intuitively, we ask that any two points of the space X could be connected by a path (a curve in X). More precisely:

Definition 1. A path in X is a continuous map $\gamma : [0, 1] \to X$. If $\gamma(0) = x$, $\gamma(1) = y$, we say that γ connects x and y, and that x and y are the endpoins of the path.

Technically speaking, we can consider paths $\gamma : [a, b] \to X$ whose domain is an arbitrary closed interval. (Reparameterizing, we can get a path $\gamma(\frac{t-a}{b-a})$ whose domain is $[0, 1]$. The reparameterized path is continuous because the expression in parantheses is contunuous, and the composition of continouous functions is continuous.)

Definition 2. A topological space X is path-connected if every two points of X can be connected by some path.

Path-connectedness is not hard to check for many subsets of a Euclidean space. In many situations, one could connect points by a straight segment or a broken line or some other reasonably simple curve. For example, a curve $\gamma(t) = (\cos t, \sin t)$ travels along the unit circle and will connect two given points if you choose its domain accordingly. For us, path-connectedness is useful because of the following:

Theorem 3. *If a topological space is path-connected, then it is connected.*

WARNING: the converse is not true! There are weird spaces that are connected but you cannot make paths.

Proof. Suppose X is path-connected but not connected, ie $X = A \cup B$, where A and B are non-empty, disjoint, and open. Pick $a \in A$, $b \in B$, and find a path $\gamma : [0, 1] \to X$ connecting a and b. This path breaks down into a part lying in A and a part lying in B, which implies that the interval is disconnected. More precisely, we can write $[0, 1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$. (The set $\gamma^{-1}(A)$ consists of all t's such that $\gamma(t)$ lies in A.) The sets $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are both open (as preimages of open sets under continuous function), they are disjoint and non-empty (why?). This means $[0, 1]$ is disconnected, a contradiction. $\square$

2. Topological invariance

Theorem 4. *Suppose X is connected, and $f : X \to Y$ is a continuous function which is onto. Then Y is connected.*

Remark 2. If f is not onto, Y may be disconnected. However, we can switch to the function $f : X \to f(X)$ which is onto (and still continuous), and conclude that $f(X)$ is connected. In words, the theorem says that a continuous image of a connected set is connected.

Proof. Suppose Y is not connected, $Y = A \cup B$, where A and B are non-empty, disjoint, and open. Then $X = f^{-1}(A) \cup f^{-1}(B)$. These two sets are open (as preimages of open sets under a continuous function); they are disjoint (no point goes to both A and B since A and B are disjoint) and non-empty (because f is onto). Then X is not connected, a contradiction. $\square$

Theorem 5. *Suppose X and Y are homeomorphic topological spaces. Then X and Y are either both connected or both disconnected.*

Proof. This follows at once from the preceding theorem. (Recall that a homeomorphism is a continuous bijection whose inverse is also continuous.) Or give a direct proof. $\square$

3. Connected components

Asking if a space is connected is basically asking whether it consists of one piece. If the space is not connected, it makes sense to ask how many pieces it has.

WARNING: the definition below ONLY deals with the situation when X has FINITELY MANY pieces (connected components). This is not always true - there spaces with infinitely many components (think $\mathbb{Q}$, for example). However, in the infinite case a different definition must be given. It is technically a lot more involved, so we avoid the infinite case.

Definition 3. Suppose the space X is the *finite* union $X = X_1 \cup X_2 \cup \cdots \cup X_k$ of non-empty subsets such that

1) each X_i is connected; 2) each X_i is both open and closed in X. 3) all X_i are disjoint, ie no two intersect.

Then we say that X_1, X_2, ...X_k are the connected components of the space X.

Note that a space with two or more connected components is disconnected. (Why? what are the "A" and "B" from the definition?)

Remark 3. In fact we can just request that all X_i's be open, and the closedness will follow (why?)

To identify connected components, you should find the corresponding subsets and carefully check all required properties (that they are disjoint, non-empty, connected and open and closed in X).

Theorem 6. *Suppose that X and Y are homeomorphic spaces. Then X and Y have the same number of connected components. In fact, any homeomorphism between X and Y must match components of X and components of Y.*

Proof. Let $f : X \to Y$ be a homeo.

Let X_0 a connected component of X. We will show that X_0 is mapped into a single component of Y (ie its image $f(X_0)$ cannot intersect two or more components). Indeed, suppose $f(X_0)$ intersects components $Y_1, Y_2, \ldots, Y_k$. then $X_0 = (f^{-1}(Y_1) \cap X_0) \cup (f^{-1}(Y_2) \cap X_0) \cup \cdots \cup (f^{-1}(Y_k) \cap X_0)$. Each of the sets in the union is relatively open in X_0 (why?), they are disjoint since Y_i's are disjoint, and all are non-empty by our assumption. This implies that X_0 is not connected, which is a contradiction.

Now run the above argument for $f^{-1} : Y \to X$. It follows that each component of Y is mapped under f^{-1} to a single component of X. This means that no two components of X could be mapped by f into the same component of Y. Thus, each component of X goes to some single component of Y, different components of X go to different components of Y, and every component of Y must be hot (because f is onto). This means that f gives a bijection between components of X and those of Y, so the two spaces must have the same number of components. □

4. Applications

With connectedness, we have developed our first *topological property*, ie a property (unlike size or shape) that remains unchanged under homeomorphisms. This gives us a way to detect when two topological spaces are *not* homeomorphic. Theorems 5 and 6 provide the basic idea. It can be pushed further if we remove a point (or several) and look at its complement in our space:

Observation. Suppose the spaces X and Y are homeomorphic, via some homeomorphism f. Pick a point $x \in X$. Then $X \setminus \{x\}$ is homeomorphic to $Y \setminus f(x)$, ie the complements of this point in X and its image in Y are homeomorphic. (Note that this uses that f is bijective, ie no other points in X go to $f(x)$.)

Corollary 2. *A closed interval and an open interval are not homeomorphic.*

Proof. Suppose $[a, b]$ is homeomorphic to (c, d). Notice that $[a, b]$ remains connected after removal of an endpoint, but (c, d) becomes disconnected no matter which point you remove. More precisely, if $f : [a, b] \to (c, d)$ is a homeomorphism, then f gives a homeo between $(a, b]$ and $(c, f(a)) \cup (f(a), d)$. The first of these is connected, the second isn't (why?), a contradiction. ☐

Corollary 3. *A circle is not homeomorphic to any interval.*

Proof. An interval becomes disconnected if you remove its midpoint. The circle remains connected if you remove an arbitrary point. (Indeed, a circle with a point removed is homeomorphic to an open interval. Intuitively, this follows by "unbending" the interval. For a precise proof, you can consider the bijection between $(0, 1)$ and the unit circle, given by $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. This function is clearly a bijection, but why a homeomorphism? First, you need to understand the topology on the circle. This is a topology of relatively open sets: the circle is a subset of $\mathbb{R}^2$. The basis is given by relative neighborhoods; those are the intersections of neighborhoods in $\mathbb{R}^2$ (ie open disks) with the circle. But these intersections are just open arcs in the circle! It is now clear that the standard basis of the topology on (0,1) - all possible open intervals in (0,1) - is sent to the basis on the circle via the map above. This means that we have a homeomorphism. ☐

We conclude by giving an application that uses a connected components count.

Corollary 4. *The block capitals X and T (considered as subsets of $\mathbb{R}^2$) are not homeomorphic.*

Proof. Idea: remove the center point in X. You get 4 connected components. Removing a point in T, you never get more than 3 components, so X and T cannot be homeomorphic.

Now let's make this idea precise. For concreteness, represent X as the set $\{(x, y) : y = \pm x, -1 \leq x, y \leq 1\}$, and T as $\{(x, 0) : -1 \leq x \leq 1\} \cup \{(0, y) : -1 < y < 0\}$. (One can wonder what happens if the letters are represented in a slightly different way; it is not hard to check that any two "reasonable" block capital shapes of the same letter are homeomorphic, but we will not worry about this.)

We need to understand the topology on X and T. Again, this is the subspace topology; relative neighborhoods are given by intersections of open disks in $\mathbb{R}^2$ with the letter. This means that the junction point of X has x-shaped neighborhoods, junction of T has T-shaped neighborhoods, and for a point inside a leg of X or T, a (small) neighborhood is given by an open interval inside that leg. (What are the neighborhoods of the endpoints of the legs?)

We now claim that $X \setminus \{(0, 0)\}$ has 4 connected components. Indeed, these components are the four intervals lying in the four quadrants of $\mathbb{R}^2$: $X_1 = \{(x, y) : y = x, 0 < x \leq 1\}$, $X_2 = \{(x, y) : y = -x, -1 \leq x < 0\}$, $X_3 = \{(x, y) : y = x, -1 \leq x < 0\}$, $X_4 = \{(x, y) : y = \pm x, -1 \leq x, y \leq 1\}$.

Each X_i is homeomorphic to a half-open interval and therefore is connected. (To see the homeomorphism, you can appeal to the general fact that geometric congruence implies homeomorphism - never mind we never checked that. For the specific case above, the projection of diagonal intervals to the corresponding intervals on the x-axis will give a homeopmorphism: it is clear that projection is bijective (for each X_i), and continuity of the projection and its inverse follows from the fact that it respects the basis of topology (the basis is given by open intervals in each case).).

Each X_i is both relatively open and relatively closed in $X \setminus \{(0, 0)\}$. You can see this by analyzing relative neighborhoods, or (easier) by noticing that $X_1 = X \setminus \{(0, 0)\} \cap \{x > 0, y > 0\}$, and the quadrant $\{x > 0, y > 0\}$ is open. (Similarly for closed and for other X_i's).

Now, suppose that X and T are homeomorphic, via some homeomorphism $f : X \to T$. Then, if we remove the point $f(0, 0)$ from T, T must break into 4 connected components. But a quick analysis of cases shows that T has no such point. Indeed, if we remove the junction point, there will be 3 components (proof similar to the case of X above). If we remove a point inside a leg, there will be two components (an interval and a T-shaped piece). Finally, if we remove an endpoint of a leg, there will be a single T-shaped component. (As before, you need to show that each of the components is both open and closed. This is similar to the case of X. You also need to show that the T-shaped piece is connected. For this, use Theorem 2, as T is the union of the horizontal interval and the vertical interval.) $\square$