MAT 364 Topology

Problem Set 4

Solutions

Problem 1. Are the following functions continuous? Are they homeomorphisms? Explain.
(a) $f : [-1, 1] \to [0, 1]$, $f(x) = |x|$
(b) $g : [0, 1) \to C$, where $C = \{(x, y) : x^2 + y^2 = 1\}$ is the unit circle in the plane, and the function g sends $x \in [0, 1)$ to the point on the circle corresponding to the angle $2\pi x$.
(If you want a formula, $g(x) = (\cos 2\pi x, \sin 2\pi x)$).
Although we discussed the second example in class, please write a detailed explanation for your answers.
Solution. (a) f cannot be a homeomorphism because it is not injective, but it is continuous. To show continuity, we'll check that $f^{-1}(D(y, r) \cap [0, 1])$ is open for every $y \in [0, 1], r > 0$. (In other words, we are checking that the inverse image of a relative neighborhood of eac point is open; the the function is continuous by the "in-between" definition of continuity that we discussed in class.) Now, if $y \in (0, 1)$, and r is small, $f^{-1}(D(y, r) \cap [0, 1]) = f^{-1}(D(y, r)) = (-y-r, -y+r) \cup (y-r, y+r)$ is open. If $y = 0$, $f^{-1}(D(0, r)) = (-r, +r)$ is open. If $y = 1$, $f^{-1}(D(1, r) \cap [0, 1]) = f^{-1}((1-r, 1]) = [-1, -1+r) \cup (1-r, 1]$ which is relatively open in $[-1, 1]$. (One can also argue from the $\epsilon - \delta$-definition).

Problem 2. We know from claculus/analysis that the following functions $f, g : \mathbb{R} \to \mathbb{R}$ are not continuous. Show this using the topological definition of continuity ("f^{-1}(open) is open").
(a) $f(x) = \begin{cases} 3x, & x \neq 2 \\ 0, & x = 2 \end{cases}$
(b) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
Solution. (a) Consider the open interval $A = (-3, 3)$, then $f^{-1}(A) = (-1, 1) \cup \{2\}$. This set is not open (2 is not an interior point) thus the function can't be continuous.
(b) the interval $(-1, 1)$ is open, but its inverse image $f^{-1}(-1, 1) = (-\infty, -1) \cup \{0\} \cup (1, \infty)$ is not (0 is not an interior point).

Problem 3. Let $A \subset \mathbb{R}^m, B \subset \mathbb{R}^n$ be some spaces. Recall that A is homeomorphic to B if there exists a function $f : A \to B$ which is a homeomorphism.
(a) Prove that if A is homeomorphic to B, then B is homeomorphic to A.

(b) Prove that if A is homeomorphic to B, and B is homeomorphic to C, then A is homeomorphic to C.
(Please give a careful argument; saying that A and B are "topologically the same" is not a proof.)
Solution. (b) If $f : A \to B$, $g : B \to C$ are homeomorphisms, then $g \circ f : A \to C$ will be a homeomorphism. Indeed, $g \circ f$ is bijective as composition of two bijections (check this if you don't remember it from MAT 200). A composition of two continuous functions is continuous by an exercise from previous homework, so $g \circ f$ is continuous. Finally, the inverse function $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is continuous, since f^{-1} and g^{-1} are continuous.
(a) is similar but easier.

Problem 4. Prove that our definition of connectedness is equivalent to Definition 2.27 (ie do exercise 2.29).
Solution. We need to prove:
X cannot be represented as $X = A \cup B$, where A, B open, non-empty, disjoint if and only if whenever $X = A \cup B$, A, B non-empty, dijoint, A or B has to contain a limit point of the other set.
This is equivalent to proving that
$X = A \cup B$, for some A, B open, non-empty, disjoint $\iff$ $X = A \cup B$ for some A, B non-empty, disjoint, such that A and B contain no limit points of one another.
We will show that the sets A, B that work for the first part work for the second part, and vice versa.
Indeed, suppose that $X = A \cup B$, for some A, B open, non-empty, disjoint. Let's prove that A contains no limit points of B. Indeed, if $x \in A$, then since A is open, x is interior for A, ie has a neighborhood consisting entirely of points of A. But then x cannot e a limit point for B, because a limit point for B is required to contain points of B in every neighborhood.
Conversely, suppose $X = A \cup B$ for some A, B non-empty, disjoint, such that A and B contain no limit points of one another. Let's prove that A and B are open. Indeed, pick $x \in A$, and check that x is interior: we need to show that there is a neighborhood of x contained entirely in A. But if there's no such neighborhood, then every neighborhood of x contains points from B, which means that $x \in A$ is a limit point for B. This is a contradiction (A contains no limit points of B), so x is interior for A.

Please also do question 2.31 from the book.
Solution. For example:
(a) A = top half-circle of a circle, B = bottom half-circle, $A \cap B$ = two points
(b) $A = [0, 3]$, $B = [1, 2]$, $A - B = [0, 1) \cup (2, 3]$.
(c) $A = [0, 1] \cup [2, 3]$, $B = [1, 2] \cup [3, 4]$, $A \cup B = [0, 4]$.