MAT 364 Topology

Problem Set 3

Solutions

2.19. Show that every set A is both open and closed relative to itself.

Proof. The question is almost trivial if you understand what it says. There are several ways to solve it. Let's look carefully.
OPEN: by definition, we need to show that every point of A is interior relative to A, i.e. that every $x \in A$ has a relative neighborhood entirely contained in A. But a relative neighborhood is a set of the form $N_x = D(x, r) \cap A$; because we intersect with A, any relative neighborhood of any point is always contained in A.
CLOSED: by definition, we need to check that every point in the complement of A in A is exterior. But $A - A = \emptyset$, so there are no points for which we'd have to check the exterior condition. Logic tells us that this condition is then satisfied.
Another solution would be to remember that relatively open sets in A are exactly sets of the form $A \cap O$, where O open in $\mathbb{R}^n$, and similarly relatively closed sets in A are of the form $A \cap C$, C closed in $\mathbb{R}^n$ (Thm 2.14). Since $A = A \cap \mathbb{R}^n$, and $\mathbb{R}^n$ is both open and closed, the statement follows. $\square$

2.20. Show that the empty set is both open and closed relative in any A.

Proof. This is similar to 2.19 – in fact can be derived from 2.19 using 2.22 (how?). Again, a proof from neighborhoods and a proof from 2.14 are possible. The proof from 2.14 uses the fact that $\emptyset = A \cap \emptyset$, and the empty set is both open and closed in $\mathbb{R}^n$. The neighborhoods proof is below.
OPEN: need to check that every point in the empty set is interior rel A. There are no points, thus nothing to check.
CLOSED: need to check that every point in $A - \emptyset = A$ is exterior relative to A. This means that every point $x \in A$ comes with a relative neighborhood N_x contained in A. But as in 2.19, every N_x is contained in A since $N_x = D(x, r) \cap A$. $\square$

2.22. A open relative to X iff $X - A$ closed rel to X.

Proof. Again we can argue from neighborhoods or from 2.14. Using 2.14 is perhaps quicker: A open rel $X \iff A = X \cap O$ for some O open in $\mathbb{R}^n$ $\overset{(*)}{\iff}$ $X - A = X \cap C$ for some C closed in $\mathbb{R}^n \iff X - A$ closed in X. To see why $(*)$ is true, set $C = \mathbb{R}^n - O$, and use the fact that C closed iff O open (in $\mathbb{R}^n$!). $\square$

2.25. A composition of two continuous functions is continuous.

Proof. One can argue from $\epsilon - \delta$ definition, but open sets give a neater proof. Consider continuous functions

$$f : A \to B, \qquad g : B \to C$$

Let U be an arbitrary open set in C, then $V = g^{-1}(C)$ is open in B. But then, since f is continuous, $f^{-1}(V) = f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(U)$ is open. It follows that $(g \circ f)$ is continuous.

$\square$