Problem 1. Recall that in a metric space $X$, we introduced notation $D(a, r) = \{ x \in X : d(x, a) < r \}$ for the open ball of radius $r > 0$ centered at $a \in X$, and $\bar{D}(a, r) = \{ x \in X : d(x, a) \leq r \}$ for the closed ball of radius $r$ centered at $a$.

(a) Prove that $\bar{D}(a, r) = \{ x \in X : d(x, a) \leq r \}$ is closed in $X$ (in the topology induced by the metric $d$).

(b) Give an example of a metric space where the open ball $D(a, r)$ is also closed, and $\bar{D}(a, r)$ is not the closure of $D(a, r)$! **Hint:** consider the metric that induces discrete topology, i.e., set $d(x, y) = 1$ for any two distinct points $x, y \in X$.

Problem 2. Let $X$ be a metric space. Suppose that a subset $A \subset X$ is compact (in the topology induced by the metric).

(a) Prove that $A$ is bounded. (This means that for a fixed point $x_0 \in X$ there exists a number $M$ such that the distance from all points of $A$ to $x_0$ is less than $M$.)

(b) Prove that $A$ is closed. **Hint:** arguing by contradiction, suppose that $a \notin A$ is not an exterior point. Cover $A$ by sets of the form $X - \bar{D}(a, r)$, where $r > 0$ (you will need to check that these sets are open). Does this open cover have a finite subcover?

Problem 3. In Homework 7, you were supposed to prove that the set $\mathbb{R}^2$ with the distance $d(x, y)$ defined by the formula

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|,$$

$x = (x_1, x_2), y = (y_2, y_2)$

is a metric space. Therefore, we can consider the topology on $\mathbb{R}^2$ that is undiced by this metric. Is this topology the same as the standard Euclidean topology on the plane, or is it different? Prove your answer.