MAT 364 Topology

Problem Set 8

due Wednesday, November 3

Problem 1. Recall that in a metric space X, we introduced notation $D(a, r) = \{x \in X : d(x, a) < r\}$ for the open ball of radius $r > 0$ centered at $a \in X$, and $\bar{D}(a, r) = \{x \in X : d(x, a) \leq r\}$ for the closed ball of radius r centered at a.

(a) Prove that $\bar{D}(a, r) = \{x \in X : d(x, a) \leq r\}$ is closed in X (in the topology induced by the metric d).

(b) Give an example of a metric space where the open ball $D(a, r)$ is also closed, and $\bar{D}(a, r)$ is *not* the closure of $D(a, r)$! **Hint:** consider the metric that induces discrete topology, ie set $d(x, y) = 1$ for any two distinct points $x, y \in X$.

Problem 2. Let X be a metric space. Suppose that a subset $A \subset X$ is compact (in the topology induced by the metric).

(a) Prove that A is bounded. (This means that for a fixed point $x_0 \in X$ there exists a number M such that the distance from all points of A to x_0 is less than M.)

(b) Prove that A is closed. **Hint:** arguing by contradiction, suppose that $a \notin A$ is not an exterior point. Cover A by sets of the form $X - \bar{D}(a, r)$, where $r > 0$ (you will need to check that these sets are open). Does this open cover have a finite subcover?

Problem 3. In Homework 7, you were supposed to prove that the set $\mathbb{R}^2$ with the distance $d(\mathbf{x}, \mathbf{y})$ defined by the formula

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|, \qquad \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_2, y_2)$$

is a metric space. Therefore, we can consider the topology on $\mathbb{R}^2$ that is undiced by this metric. Is this topology the same as the standard Euclidean topology on the plane, or is it different? Prove your answer.