A BRIEF INTRODUCTION TO LOBACHEVSKIAN GEOMETRY

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Abstract. These are lecture notes for MAT 360, Spring 2011.

1. A bit of history and discussion of axiomatics

Euclid's famous "5th postulate" states that, given a line l in the plane and a point A not on l, there exists a *unique* line passing through A and parallel to l. The point of the postulate is that we can never have two different lines with this property. Indeed, it is not hard to prove (using other axioms) that a line through A that is parallel to l exists; such a line can be constructed with a compass and straightedge.

What exactly does it mean that the 5th postulate cannot be derived from other axioms? People certainly tried to prove the uniqueness of the parallel line - the statement looks like it could be a theorem. (Recall, for example. that uniqueness of a perpendicular from a given point to the given line can be easily proved.) However, no-one was able to prove this uniqueness of parallels. Does this mean that it cannot be proved? Not really: perhaps the proof was just too difficult and elusive for anyone to discover. To demostrate that the 5th postulate cannot be proved, one really needs to show that it is "independent" from the other axioms; indeed, one needs to demonstrate that there exists a "geometry" that satisfies all the other axioms (such as, through any two points there is a unique line, the existence of certain isometries, etc) but not the uniqueness of parallels. Such a "geometry" was first constructed by Lobachevsky in early 19th century; the construction he came up with is now called the Lobachevskian plane.

Before delving into the details of the construction, let us discuss what is meant by "geometry" and by the lack of uniqueness of parallels. Does it mean that in an alternative universe, one could suddenly draw a bunch of different lines through A, and none of those lines would intersect l? In a sense, yes – but the "lines" and "points" might look quite different from those in Euclidean geometry. To understand this better, let us consider the axioms of the arithmetics of numbers. Those are very familiar - one can add and multiply integer numbers, and they satisfy certain properties, such as $a+b=b+a$, $a(b+c)=ab+ac$, $(a+b)c=ac+bc$, $a+(-a)=0$, $ab=ba$, etc. If one accepts the rest of the properties as axioms, can the last one, $ab=ba$, be proved as a theorem? To show that this property is independent from other axioms, one needs to construct some alternative version of "numbers" that satisfy all the other axioms, but not $ab=ba$. Again, this doesn't mean that in alternative universe, the multiplication table goes wrong, and $5\cdot 3\neq 3\cdot 5$; rather, "numbers" can be some more involved objects. For example, if the inhabitants of the alternative universe are working with matrices instead of numbers, this will be exactly the situation described above. (Recall that $n\times n$ matrices can be added and multiplied, and have a lot of nice properties, such as $A+B=B+A$, etc; however, in general AB and BA are different matrices.)

2. The Poincaré model

Let us now describe one possible construction of the Lobachevskian geometry, and set up a "dictionary" that will tell us what kind of things will play the roles of the plane, lines, points, angles, isometries, etc. This particular construction is called the Poincaré model (or the Poincaré plane).

2.1. **The plane, points and lines.** Fix a circle C_0 in the plane, centered at some point O, and let D be the disk that it bounds. The disk D will be our "plane" for the non-Euclidean geometry; the "points" are the points in D (not including the boundary circle C_0). The "lines" will be circles that are orthogonal to C_0 (or rather, the arcs of these circles that lie in D), together with all diameters for C_0. Here and below, the angle between two circles (or the angle between a circle and a line) is understood to be the angle between the tangent lines at the intersection point.

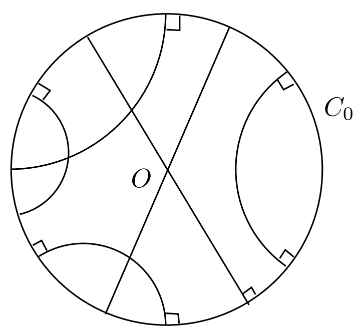

Figure 1. The Poincaré model.

Before moving on, let's make a couple of observations about orthogonal circles.

Lemma 1. *Let c be a circle orthogonal to C_0, and let A and B denote the intersection points of circles C_0 and c. Then the tangent lines to c at A and B intersect at O; similarly, the tangent lines to C_0 at A and B intersect at the center of c.*

Proof. Recall that the radius OA of the circle C_0 is perpendicular to the tangent line t to C_0 at A. Since the tangent line l to c at A is also perpendicular to c (the circles are orthogonal), l contains the radius OA and thus goes though O. Similarly, the tangent line m to c at B contains the radius OB, so l and m intersect at O. (The tangents to the circle C_0 are treated in exactly the same way.) See Figure 2. □

Lemma 2. *None of the circles orthogonal to C_0 pass through O.*

Proof. This is a corollary of the previous lemma. Indeed, if c is a circle orthogonal to C_0, its tangents (at the intersection points with C_0) meet at O, and thus this point lies outside of the circle. □

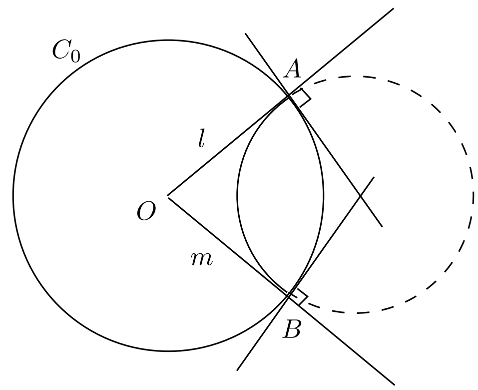

FIGURE 2

These two lemmas give us a better understanding of the Poincare disk picture. Indeed, when the radius of the circle c gets larger and larger, this circle gets closer and closer to O, and its arc inside D looks more and more like a straight segment; that is, very close to O "lines" look almost like Euclidean lines, and closer to C_0 they become more curved. Notice also that any diameter of C_0 is perpendicular to C_0 (and no other chords have this property). Thus we can say that the "lines" in the Poincaré plane are the lines and circles orthogonal to C_0.

2.2. **Isometries.** Now let's try to see what could be meant by "isometries" in this setup. We should expect "isometries" to map the "plane" to itself, and to map "lines" to "lines". Clearly, rotations about O and reflections through any diameter have these properties. More interestingly, inversions through circles orthogonal to C_0 also work: we know that they send lines/circles to lines/circles and preserve angles, so they have to map lines/circles perpendicular to C_0 to lines/circles perpendicular to C_0. We've shown that these special inversions map "lines" to "lines". Now let's check that they map the "plane" D to itself. Indeed, if c is orthogonal to C_0 and intersects it at A and B, and I_c is the inversion through c, the image of C_0 is 1) a circle (it can't be a line because C_0 doesn't go through the center of c), 2) orthogonal to c, 3) intersects c at the same points A and B. By Lemma 1, the center of the image of C_0 must be at O, and this implies that C_0 is mapped to itself. Now we are ready to state that:

The "isometries" of the Poincaré plane are taken to be all reflections through diameters of C_0, inversions through circles orthogonal to C_0, and arbitrary compositions of (any number of) such reflections and inversions.

Note that rotations about O can be written as compositions of reflections. One can do a detailed study of compositions of reflections/inversions (similar to what we did in the Euclidean case), and show that all "isometries" have fairly simple form. We will not pursue this.

Theorem 3. *Given two arbitrary points A, B in the Poincaré plane, two arbitrary "rays" l'_A, l'_B with vertices at these points, and two chosen "half-planes" H_A, H_B cut off by the* *"line" l_A resp. l_B, there exists a unique isometry I mapping A to B, l'_A to l'_B, and H_A to H_B.*

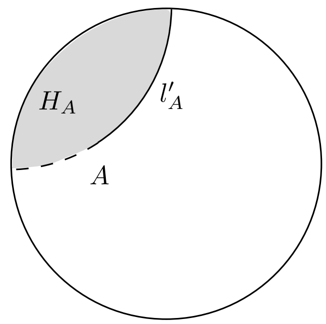

FIGURE 3

Proof. First, let's see what "rays" and "half-planes" mean in our language. As always, a "ray" is a part of a "line" cut off by a point; since a typical line is an arc orthogonal to C_0, a "ray" is half-arc (see Figure 3). Similarly, a "line" divides the "plane" into two "half-planes", so "half-planes" must be the curved regions as shown.

For the existence part of the proof, it suffices to find the required "isometry" for the case where one of the points is O, the center of C_0. Indeed, if I_A is an "isometry" taking A to O, l'_A to some given l'_O, and H_A to some given H_O, and I_B is a similar "isometry" for B, then $I_B^{-1} \circ I_A$ is an "isometry" taking A to B, l'_A to l'_B, and H_A to H_B. (Why?) The case when one the points is O is easier because then l'_O is a radius and H_O is a half-disk of D cut off by the corresponding diameter. So if we find an "isometry" taking A to O, we can then match the rays by a rotation at O, and the half-planes by a reflection though a diameter. (Make sure you understand this!)

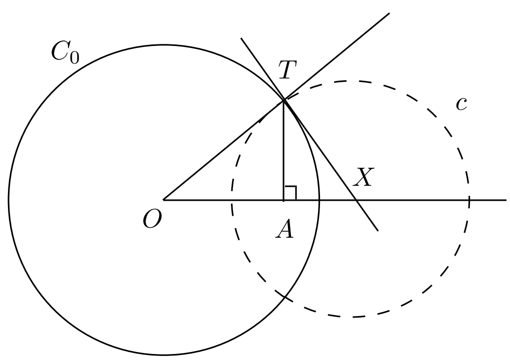

FIGURE 4

It remains to find an "isometry" taking A to O; indeed we'll find an inversion with this property. The center of such inversion must lie on the ray OA (why?); we need to locate its center X. The point X can be constructed as follows. Raise a perpendicular to OA at

A, and let T be its intersection with C_0. Draw a tangent line to C_0 at T; then X is the intersection of this tangent and the line OA. (Figure 4.) Let c be the circle centered at X with radius XT; then c is orthogonal to C_0 (this is the converse to Lemma 1). Since the triangles $\triangle OTX$ and $\triangle TAX$ are similar (why?),

$$\frac{OT}{XT} = \frac{XT}{XA},$$

so $|XA| \cdot |XT| = |XT|^2$. It follows that the inversion through c maps O to A.

We will not prove the uniqueness part of the theorem as this requires classification of isometries. □

2.3. **Congruent figures.** As in Euclidean geometry, we will call two figures "congruent" if they can be matched by an "isometry". In particular, one can talk about "congruent segments" - typically those will be arcs of circles perpendicular to C_0 and appear to be of different (Euclidean) size. However, they are congruent in our sense whenever one can be transformed into another by an "isometry" (such as inversion).

2.4. **How about distance?** Recall that the word *isometry* is supposed to refer to distance-preserving maps. This seems to be a contradiction since inversions clearly don't preserve the Euclidean distance: the inversion through a circle c of a small radius swaps the large region outside of c and the small region inside. However, this only means that the Euclidean distance is not the right one to use in our setup. Indeed, one can introduce a "distance" in the Poincaré plane (given by a complicated formula) that is preserved by the "isometries" and satisfies the triangle inequality and other natural properties. If two (non-Euclidean) "segments" are "congruent", the "distance" between their endpoints will be the same. We will not discuss distance in further detail.

2.5. **Angles.** The "angles" turn out to be simple in our setup: they are just the Euclidean angles (measured between lines and/or circles by considering tangents when necessary.) Notice that just as one would expect, all the "isometries" preserve "angles" (because the Euclidean angles are preserved by inversions and reflections.) One could also compare two angles, by matching their vertices and lining up one of the sides by an "isometry".

In the non-Euclidean context, one could establish a lot of familiar theorems: the congruence tests for triangles, the statement that opposite the greater "angle" in a triangle lies a greater "side", etc. It is a very good exercise to think back to the familiar Euclidean proofs and repeat them for the Poincaré model.

3. Axioms

To call our construction a geometry, we need to check that the necessary axioms are satisfied. We have already seen a fundamental property of isometries (Theorem 3 above). Now, let us prove

Theorem 4. *Through any two "points" in the Poincaré "plane" there exists a unique "line".*

Before turning to the proof, let's see what's happening here. Are we trying to prove an axiom?! Not quite – we are trying to prove that a particular (fairly weird!) construction has the same properties as we expect in the Euclidean geometry. We will translate the statement above to the Euclidean terms: the theorem says that through any points A, B in D one can draw a segment of the diameter or an arc of a circle orthogonal to C_0 (but not both), and that such a segment or an arc is unique.

Studying arcs of orthogonal circles is not that easy; instead, we will use "isometries" for a helpful trick. Indeed, by Theorem 3 we can find an "isometry" I that maps the point A to O (the center of C_0), and B to some other point X. Note that I would map any "line" through A and B to a "line" through O and X, and that the inverse "isometry" I^{-1} (why does it exist?) maps any "line" through O and X to a "line" through A and B. Thus, to prove existence and uniqueness of a "line" through A, B, it suffices to prove the existence and uniqueness of a line through O, X. The latter is easy: since O is the center, there is a unique line OX which is a segment of the diameter. There can be no arcs of othogonal circles though O, X, since none of those circles passes through O (Lemma 2).

Corollary 5. *Any two distinct "lines" intersect at one point at most.*

Proof. The proof is exactly the same as in Euclidean geometry (except that the meaning of the word "line" is different): if two distinct "lines" met at two points A and B, there would be two different lines through A, B, which contradicts the previous theorem. □

Many basic theorems in the Poincaré model can be proved by repeating the corresponding proofs in Euclidean geometry (with a different interpretation of all relevant terms). For example:

Theorem 6. *Given a "line" l and a point A not on l, there exists a unique "perpendicular" dropped from A to l.*

Proof. Let's use the strategy from Euclidean geometry: consider the point A' symmetric to A about the "line" l, draw the "line" m through A and A', and use symmetry to show that m and l intersect at the right angle. Here, if the "line" l is a circle in Euclidean geometry, the symmetry about l is to be understood as inversion through the corresponding circle. Similarly, the "line" m through A and A' may be a line or a circle; the symmetry (together with uniqueness of the "line" through A, A') implies that m goes to itself under the inversion through l. Then it is not hard to see that m must be orthogonal to l.

By a similar reasoning, if there were two perpendiculars from A to l, one could use symmetry to construct two "lines" through A, A'.

(This is only a sketch of the proof; working out the details is a useful exercise.) □

The notion of "parallel" lines can be introduced in our setup as well: two "lines" are called parallel if they don't meet. As in Euclidean geometry, one can show that for any given "line" l and point A not on l there is a line m passing through A and parallel to l: drop a "perpendicular" p from A to l, construct a "line" m perpendicular to p and passing through A. Then l and m are parallel: if they met at any point, there would be two perpendiculars (l and m) from that point to p.

Uniqueness of the parallel line (for given l, A) is strikingly absent. (Recall that this is the whole purpose of our model of non-Euclidean geometry.)

Theorem 7. *Through any point A there exist infinitely many "lines" parallel to the given line l.*

Proof. See Figure 5. □

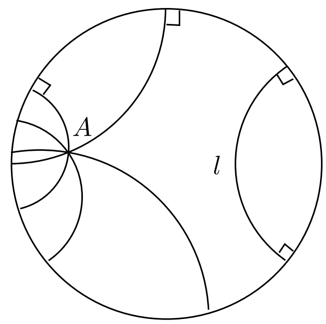

FIGURE 5. The lines parallel to l.

4. The sum of angles in a triangle

In Euclidean geometry, the sum of angles in any triangle is 180 degrees. This is a corollary of the 5th postulate. Since the 5th postulate is no longer true in non-Euclidean geometry, we shouldn't expect the sum of angles to automatically remain the same – we certainly cannot "translate" the Euclidean proof into our context. It turns out that:

Theorem 8. *The sum of angles of any "triangle" in the Lobachevskian plane is strictly less than the straight angle.*

Proof. Notice that "triangles" have sides that are segments of "lines" - ie segments of diameters or arcs of circles orthogonal to C_0.

Since "isometries" do not change angles, we can apply an "isometry" sending one of the vertices of a given triangle to the center O of C_0. Thus, it suffices to show that any "triangle" OAB has a sum of angles less than 180 degrees. Such a "triangle" has two straight sides meeting at O, and one curved side AB. It will be helpful to consider the Euclidean triangle with the same vertices, i.e. replace the arc AB by a straight segment. In the Euclidean triangle, the angles add up to 180 degrees. The angle O is clearly the same in the straight-sided and curved triangles. The angles A and B are smaller for the curved triangle (see Figure 6); therefore, the curved triangle has the sume of angles less than 180 degrees. □

It is interesting to notice that for small triangles near the point O, the sum of angles gets close to 180 degrees; the non-Euclidean effects become more noticeable when one approaches the coundary circle C_0.

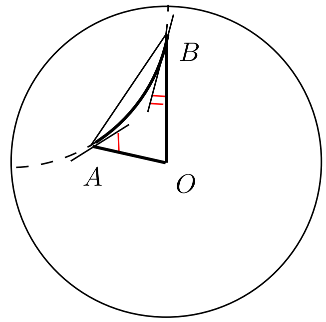

Figure 6