Problem Set 11, due Thursday, May 12

Problem 1. Prove that
(a) For any two points $A, B$ that are not symmetric about a line $l$, there exists a circle that passes through $A$ and $B$ and is not orthogonal to $c$.
(b) If the point $B$ is not the image of the point $A$ under the inversion about a circle $c$, then there exists a circle that passes through $A$ and $B$ and is not orthogonal to $c$.

In other words, if all circles passing through $A$ and $B$ are orthogonal to the line $l$ (resp. the circle $c$), then the points $A$ and $B$ are symmetric about this line (resp. this circle).

Problem 2. Prove that a composition of two inversions with the same center is a homothety centered at the same point. Find the coefficient of this homothety in terms of the degrees of the inversions.

Problem 3. In $(x, y)$ plane, let $C$ be a circle of radius 1 centered at the origin, and consider the inversion $I$ about this circle. Find the images under the inversion $I$ of
(a) the line $\{x = 2\}$
(b) the circle $c_1$ of radius $\frac{1}{2}$, centered at the point $(-\frac{1}{2}, 0)$
(c) the circle $c_2$ of radius 1, centered at $(3, 0)$.
Justify your answers.

Problem 4. (a) Consider an inversion $I$ centered at $O$, a line $l$ passing through $O$, and a circle $c$ passing through $O$. Show that the angle between the images of $l$ and $c$ under $I$ is the same as the angle between $l$ and $c$.
(b) Show that if $c_1$ and $c_2$ are two circles passing through $O$, the angle between them is the same as the angle between their images.

Problem 5. (This question will be used in the next one.)
Given the segments $AB, CD$, construct a segment $EF$ such that $|EF|^2 = |AB| \cdot |CD|$.
Hint: use similarity as in Question 4(a,b) in Homework 10.

Problem 6. Given a line $l$ and two points $A, B$ not on the line, construct a circle $C$ that passes through $A$ and $B$ and is tangent to $l$.
Hint: Draw a line $m$ through $A$ and $B$. The case where $m \parallel l$ is relatively easy. If the lines are not parallel, consider the point $M = m \cap l$. Use the degree of this point with respect to the required circle to locate the point $T$ of tangency, and Question 5 to construct $T$. Then construct (how?) a circle through $A, B, T$.
(The two construction problems above are to be solved, as usual, with a compass and straightedge. Please justify your constructions.)