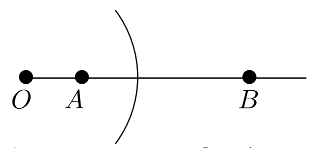
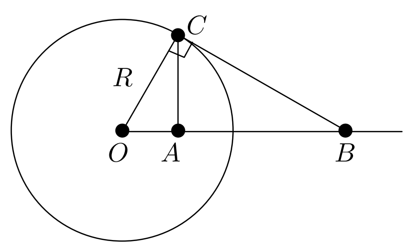
1. Inversions.

1.1. Definitions of inversion. Inversion is a kind of symmetry about a circle. It is defined as follows. The *inversion of degree R^2 centered at a point O* maps a point $A \neq O$ to the point B on the ray OA such that R is the geometric mean of OA and OB, that is $|OA||OB| = R^2$, or $|OB| = \frac{R^2}{|OA|}$.



A geometric construction relating points O, A and B looks as follows.



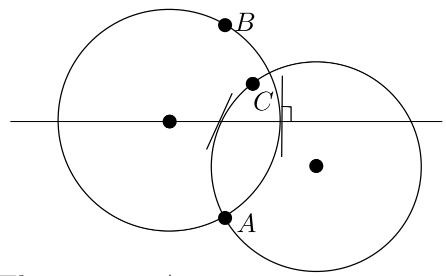
On O the inversion is not defined, and O is not the image of any point under the inversion. Thus, the inversion is really a mapping of the plane punctured at O to itself.

Observe that points of the circle centered at O of radius R are fixed under the inversion, points of the disk bounded by this circle are mapped to points outside the disk and vice versa. This circle is called the *circle of the inversion* and the inversion is referred to as the *inversion about this circle.*

The square of an inversion, that is an inversion composed with itself, is the identity map. In other words, an inversion is invertible map and the map inverse to an inversion is the same inversion.

The definitions of reflection about a line and inversion does not look similar. However these two transformations admit similar definitions.

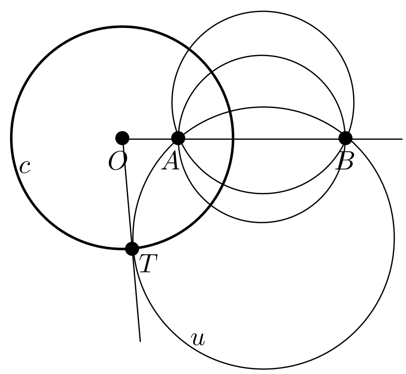
1.A. Theorem. *Any circle passing through points symmetric about a line is orthogonal to the line. For any two points non-symmetric about a line l there exists a circle passing through the points which is not orthogonal to l.*



Exercise. Prove Theorem *1.A*. □

This theorem allows to define reflection about a line l as a map which maps a point A to a point B such that any circle passing through A and B is orthogonal to l.

1.B. Theorem. *Any circle passing through a point A and its image B under the inversion about a circle c is orthogonal to c. If B is not the image of A under the inversion about a circle c, then there exists a circle passing through the points which is not orthogonal to c.*



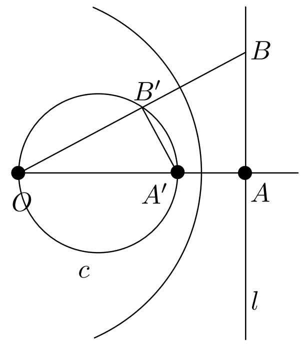
Proof. For any circle u passing through points A and B the degree of the center of inversion O with respect to u is $|OA||OB|$. Therefore $|OT|^2 = |OA||OB|$, where OT is the segment of the line tangent to u between the center of inversion and the point of tangency.

On the other hand, $|OA||OB|$ is equal to the degree R^2 of the inversion, that is to the square of the radius of circle c. Therefore $T \in c$. Hence, OT is a radius of c. As a radius of c, it is perpendicular to the tangent of c. Thus at T the lines to u and c are perpendicular to each other.

A proof of the second statement is an exercise. □

1.2. Images of lines and circles. Obviously, a line passing through the center of an inversion is mapped by the inversion to itself.

1.C. Theorem. *The image under an inversion of a line l not passing through the center O of the inversion is a circle c passing through O and having at O a tangent line parallel to l.*



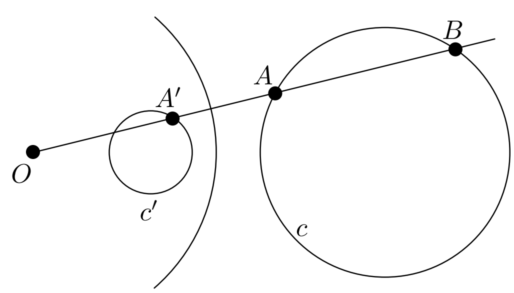
Proof. Drop the perpendicular OA to l from O. Let A be its intersection with l. Let A' be the image of A under the inversion. Take arbitrary point $B \in l$. Denote by B' its image under the inversion. By the definition of inversion $|OA||OA'| = |OB||OB'|$. Therefore $\frac{OA}{OB} = \frac{OB'}{OA'}$. By SAS-test for similar triangles, $\triangle OAB$ is similar to $\triangle OB'A'$. Therefore $\angle A'B'O = \angle OAB$. The latter angle is right, because $OA \perp l$. Hence B' belongs to the circle with diameter OA'.

Vice versa, let us take any point B' of the circle with diameter OA'. Draw a ray OB' and denote the intersection of this ray with l by B. Triangles $\triangle OB'A'$ and $\triangle OAB$ similar by the AA-test. Hence $\frac{OB}{OA} = \frac{OA'}{OB'}$ and $|OB||OB'| = |OA||OA'|$. Therefore, B' is the image of B under the inversion. □

1.D. Corollary. *The image under an inversion of a circle c passing through the center O of the inversion is a line which is parallel to the line tangent to c at O.*

Proof. This follows from Theorem *1.C*, because an inversion is inverse to itself. □

1.E. Theorem. *The image under an inversion of a circle c that does not pass through the center O of the inversion is a circle c' that is the image of c under a homothety centered at O.*



Proof. Let A be a point of circle c, and A' be the image of A under the inversion. Denote by B the second intersection point of the ray OA with c. By definition of inversion, $|OA'| = \frac{R^2}{|OA|}$, where R^2 is the degree of inversion. On the other hand, $|OA| = \frac{d^2}{|OB|}$, where d^2 is the degree of O with respect to the circle c. Recall that d does not depend on the points A and B, this is the length of segment of a tangent line from O to c between O and the point of tangency.

Substituting this formula to the formula for $|OA'|$, we get

$$|OA'| = \frac{R^2}{d^2}|OB|.$$

This means that A' is the image of B under the homothety with center O and ratio $\frac{R^2}{d^2}$. Hence, the image of c under the inversion is the image of c under this homothety. $\square$

1.F. Theorem. *A composition of two inversions with the same center is a homothety centered at the same center. The ratio of this homothety is the ratio of the degrees of the inversions.*

Proof. Exercise.

$\square$

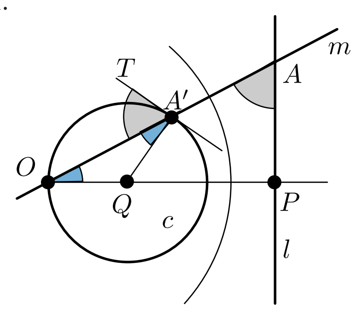
1.G. Theorem. *An inversion preserves angles between lines and circles.*

Proof. Let us begin with special cases.

The first case is the angle between two lines, l and m. Their images will be circles c_l and c_m passing through O, the center of inversion. We are interested in the angle between these circles at the point which is the image of the intersection $l \cap m$. First, let's consider the angle at the other intersection point, which is the point O.

The angle between the circles at O is by definition is the angle between their tangent lines. But the tangent line to c_l (resp. c_m) at O is parallel to l (resp. m), so the angle between tangent lines is the same as the angle between l and m. At the other intersection point - the point $l \cap m$ - the angle is the same as the angle at O by symmetry.

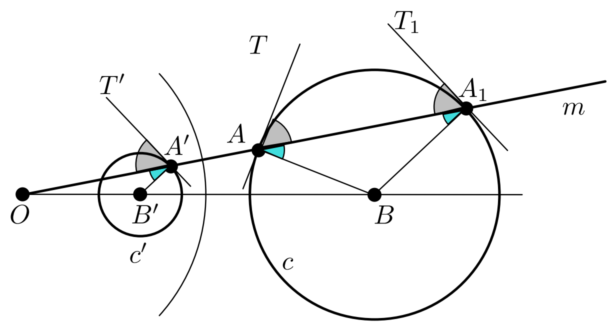
One can also prove the same fact in a different way, as follows. Consider, first, the angle between two lines, one of which passes through the center of inversion.



Let the lines be l and m, the center of inversion be $O \in m$. Then the image of m under the inversion is m, while the image of l is a circle c passing through O. The center Q of c lies on the perpendicular OP dropped from O to l. The angles $\angle AOP$ and $\angle OAP$ are complementary. The angles $\angle QA'O$ and $\angle AOQ$ are equal as angles in an isosceles triangle $\triangle OQA'$. The angle $\angle TA'Q$ is right as an angle between a radius QA' and tangent line $A'T$. Therefore angles $\angle TA'O$ and $\angle QA'O$ are complementary. Consequently, $\angle OAP = \angle TA'O$. $\square$

An angle between arbitrary two lines can be represented as the sum or difference of angles between the same lines and a line passing through the center of inversion.

Consider now the angle between a line m passing through the center O of inversion and a circle c which does not pass through O. See the picture:



Let c' be the image of c under the inversion. We know from Theorem *1.E* that c and c' are related by homothety centered at O that sends the point A_1 to A'. This homoethety sends the center of one circle to the center of the other. Thus, triangles $\triangle OA'B'$ and OBA_1 are similar, and angles $\angle OA'B'$ and $\angle OA_1B$ are congruent. But BAA' is an isosceles triangle, so $\angle OA_1B = \angle BAA_1$. We would like to show that angle between the line m and the circle c (i.e. the angle TAA_1 between m and the tangent line to c at A) is the same as the angle between the line m and the circle c' (i.e. the angle $T'A'O$ between m and the tangent line to c' at A'). But these angles complement the congruent angles $\angle OA'B'$ and $\angle BAA_1$ to right angles (since a tangent line is perpendicualr to a radius), therefore they are congruent. $\square$

The case of the angle between a line passing through O and a circle passing through O is an exercise.

To complete the proof, notice that an angle between arbitrary two circles, or an arbitrary line and an arbitrary circle, can be represented as the sum or difference of angles between the same lines or circles and a line passing through the center of inversion. Thus, the general case reduces to the special ones above. $\square$