Isometries.

Congruence mappings as isometries. The notion of *isometry* is a general notion commonly accepted in mathematics. It means a mapping which preserves distances. The word *metric* is a synonym to the word *distance*. We will study isometries of the plane. In fact, we have already encountered them, when we superimposed a plane onto itself in various ways (eg by reflections or rotations) to prove congruence of triangles and such. We now show that each isometry is a "congruence mapping" like that.

Theorem 1. *An isometry maps*
(i) straight lines to straight lines;
(ii) segments to congruent segments;
(iii) triangles to congruent triangles;
(iv) angles to congruent angles.

Proof. Let's show that an isometry S maps a segment AB to segment $S(A)S(B)$ which is congruent to AB. It is clear (from the definition of isometry) that the distance between $S(A)$ and $S(B)$ is the same as the distance between A and B. However, we need to check that the image of AB will indeed be a straight line segment. To do so, pick an arbitrary point X on AB. Then $S(A)S(B) = AB = AX + XB = S(A)S(X)+S(X)S(B)$, and by the triangle inequality the point $S(X)$ must be on the segment $S(A)S(B)$ (otherwise we would have $S(A)S(X) + S(X)S(B) > S(A)S(B)$. So image of the segment AB lies in the segment $S(A)S(B)$, and indeed, covers the whole of $S(A)S(B)$ without leaving any holes: if X' is a point on $S(A)S(B)$, find X on AB such that $XA = X'S(A)$, $XB = X'S(B)$, then $S(X) = X'$. □

Examples of isometries. We have encountered quite a few examples before: reflections, rotations, and translations are all isometries. (It is pretty easy to see that the distances are preserved in each case: for instance, a reflection R_l through the line l maps any segment AB to a symmetric, and thus congruent, segment $A'B'$.) Let's look at some examples more closely.

Translations and central symmetries. A map of the plane to itself is called a *translation* if, for some fixed points A and B, it maps a point X to a point $T(X)$ such that $ABT(X)X$ is a parallelogram. (Note the order of points!)

Here we have to be careful with the notion of parallelogram, because a parallelogram may degenerate to a figure in a line. Not any degenerate quadrilateral fitting in a line deserves to be called a parallelogram, although any two sides of such a degenerate quadrilateral are parallel. By a parallelogram we mean a sequence of four segments KL, LM, MN and MK such that KL is congruent and parallel to MN and LM is congruent and parallel to MK. This definition describes the usual parallelograms, for which congruence can be deduced from parallelness and vice versa, and the degenerate parallelograms.

Theorem 2. *For any points A and B there exists a translation mapping A to B. A translation is an isometry.*

Proof. Any three points A, B and X can be completed in a unique way to a parallelogram $ABX'X$. Define $T(X) = X'$. For any points X, Y the quadrilateral $XYT(Y)T(X)$ is a parallelogram, since $XT(X)||AB||YT(Y)$. Therefore, $XY = T(X)T(Y)$, so T is an isometry. □

Denote by T_{AB} the translation which maps A to B.

Theorem 3. *The composition of any two translations is a translation.*

Proof. Exercise. □

Theorem *3* means that $T_{BC} \circ T_{AB} = T_{AC}$.

Fix a point O. A map of the plane to itself which maps a point A to a point B such that O is a midpoint of the segment AB is called the *symmetry about a point* O.

Theorem 4. *A symmetry about a point is an isometry.*

Proof. SAS-test for congruent triangles (extended appropriately to degenerate triangles.) □

Theorem 5. *The composition of any two symmetries in a point is a translation. More precisely, $S_B \circ S_A = T_{2\overrightarrow{AB}}$, where S_X denotes the symmetry about point X.*

Proof. Exercise. □

Remark. The equality

$$S_B \circ S_A = T_{2\overrightarrow{AB}}$$

implies a couple of other useful equalities. Namely, compose both sides of this equality with S_B from the left:

$$S_B \circ S_B \circ S_A = S_B \circ T_{2\overrightarrow{AB}}$$

Since $S_B \circ S_B$ is the identity, it can be rewritten as

$$S_A = S_B \circ T_{2\overrightarrow{AB}}.$$

Similarly, but multiplying by S_A from the right, we get

$$S_B = T_{2\overrightarrow{AB}} \circ S_A.$$

Corollary. *The composition of an even number of symmetries in points is a translation; the composition of an odd number of symmetries in points is a symmetry in a point.*

Remark. In general, it is clear that a composition of isometries is an isometry: if each mapping keeps distances the same, their composition also will. It is trickier, however, to see the resulting isometry explicitly; we will prove a few theorems related to compositions of isometries. To practice with compositions, consider, for example, a reflection about a line l and a rotation by $90°$ counterclockwise about a point

$O \in l$. When composed in different order (rotation followed by reflection vs reflection followed by rotation), these yield reflections about **different** lines. The proof that the composition is a reflection can be obtained by an explicit examination of which points go where; by Theorem *6*, it suffice to examine 3 non-collinear points.

Recovering an isometry from the image of three points.

Theorem 6. *An isometry of the plane can be recovered from its restriction to any triple of non-collinear points.*

Proof. Given images A', B' and C' of non-collinear points A, B, C under and isometry, let us find the image of an arbitrary point X. Using a compass, draw circles c_A and c_B centered at A' and B' of radii congruent to AX and BX, respectively. They intersect in at least one point, because segments AB and $A'B'$ are congruent and the circles centered at A and B with the same radii intersect at X. There may be two intersection point. The image of X must be one of them. In order to choose the right one, measure the distance between C and S and choose the intersection point X' of the circles c_A and c_B such that $C'X'$ is congruent to CX. □

In fact, there are exactly two isometries with the same restriction to a pair of distinct points. They can be obtained from each other by composing with the reflection about the line connecting these points.

Isometries as compositions of reflections.

Theorem 7. *Any isometry of the plane is a composition of at most three reflections.*

Proof. Choose three non-collinear points A, B, C. By theorem *6*, it would suffice to find a composition of at most three reflections which maps A, B and C to their images under a given isometry S.

First, find a reflection R_1 which maps A to $S(A)$. The axis of such a reflection is a perpendicular bisector of the segment $AS(A)$. It is uniquely defined, unless $S(A) = A$. If $S(A) = A$, one can take either a reflection about any line passing through A, or take, instead of reflection, an identity map for R_1 .

Second, find a reflection R_2 which maps segment $S(A)R_1(B)$ to $S(A)S(B)$. The axis of such a reflection is the bisector of angle $\angle R_1(B)S(A)S(B)$.

The reflection R_2 maps $R_1(B)$ to $S(B)$. Indeed, the segment $S(A)R_1(B) = R_1(AB)$ is congruent to AB (because R_1 is an isometry), AB is congruent to $S(A)S(B) = S(AB)$ (because S is an isometry), therefore $S(A)R_1(B)$ is congruent to $S(A)S(B)$. Reflection R_2 maps the ray $S(A)R_1(B)$ to the ray $S(A)S(B)$, preserving the point $S(A)$ and distances. Therefore it maps $R_1(B)$ to $S(B)$.

Triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are congruent via an isometry $S \circ (R_2 \circ R_1)^{-1} = S \circ R_1 \circ R_2$, and the isometry is identity on the side $S(AB) = R_2 \circ R_1(AB)$. Now either $R_2(R_1(C)) = C$ and then $S = R_2 \circ R_1$, or the triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are symmetric about their common side $S(AB)$. In the former case

$S = R_2 \circ R_1$, in the latter case denote by R_3 the reflection about $S(AB)$ and observe that $S = R_3 \circ R_2 \circ R_1$. □

Compositions of two reflections.

Theorem 8. *The composition of two reflections in non-parallel lines is a rotation about the intersection point of the lines by the angle equal to doubled angle between the lines. In formula:*

$$R_{AC} \circ R_{AB} = Rot_{A,2\angle BAC},$$

where R_{XY} denotes the reflection in line XY, and $Rot_{X,\alpha}$ denotes the rotation about point X by angle α.

Proof. Pick some points whose images under reflections are easy to track. From symmetries/congruent triangles in the picture, it is clear that effect of two refections is that of a rotation. Since we know that an isometry is determined by the image of 3 non-collinear points, the ir no need to consider all possible positions of the points. □

Theorem 9. *The composition of two reflections in parallel lines is a translation in a direction perpendicular to the lines by a distance twice larger than the distance between the lines.*

More precisely, if lines AB and CD are parallel, and the line AC is perpendicular to the lines AB and CD, then

$$R_{CD} \circ R_{AB} = T_{2\overrightarrow{AC}}.$$

Proof. Similar to the above. □

Application: finding triangles with minimal perimeters. We have considered the following problem:

Problem 1. *Given a line l and points A, B on the same side of l, find a point $C \in l$ such that the broken line ACB would be the shortest.*

Recall that a solution of this problem is based on reflection. Namely, let $B' = R_l(B)$. Then the desired C is the intersection point of l and AB'.

Notice that this problem can be reformulated as finding $C \in l$ such that the perimeter of the triangle ABC is minimal.

Problem 2. *Given lines l, m and a point A, find points $B \in l$ and $C \in m$ such that the perimeter of the triangle ABC is the smallest possible.*

Idea that solves Problem 2. Reflect point A through lines l and m, that is, consider points $B' = R_l(A)$ and $C' = R_m(A)$. Use these points to find B and C (how?), and prove that the resulting triangle indeed has the smallest perimeter.

Problem 3. *Given lines l, m and n, no two of which are parallel to each other. Find points $A \in l$, $B \in m$ and $C \in n$ such that triangle ABC has minimal perimeter.*

If we knew a point $A \in l$, the problem would be solved like Problem 2: we would connect points $R_m(A)$ and $R_n(A)$ and take B and C to be the intersection points of this line with m and n. So, we have to find a point $A \in l$ such that the segment $R_m(A)R_n(A)$ would be minimal.

The endpoints $R_m(A)$, $R_n(A)$ of this segment belong to the lines $R_m(l)$ and $R_n(l)$ and are obtained from the same point $A \in l$. Therefore

$$R_n(A) = R_n(R_m(R_m(A))) = R_n \circ R_m(B),$$

where $B \in R_m(l)$. So, one endpoint is obtained from another by $R_n \circ R_m$.

By Theorem *9*, $R_n \circ R_m$ is a rotation about the point $m \cap n$. We look for a point B on $R_m(l)$ such that the segment $BR_n \circ R_m(B)$ is minimal.

The closer a point to the center of rotation, the closer this point to its image under the rotation. Therefore the desired B is the base of the perpendicular dropped from $m \cap n$ to $R_m(l)$. Hence, the desired A is the base of perpendicular dropped from $m \cap n$ to l.

Since all three lines are involved in the conditions of the problem in the same way, the desired points B and C are also the endpoints of altitudes of the triangle formed by lines l, m, n.

Composition of rotations.

Theorem 10. *The composition of rotations (about points which may be different) is either a rotation or a translation.*

Prove this theorem by representing each rotation as a composition of two reflections about a line. Choose the lines in such a way that the second line in the representation of the first rotation would coincide with the first line in the representation of the second rotation. Then in the representation of the composition of two rotations as a composition of four reflections the two middle reflections would cancel and the whole composition would be represented as a composition of two reflections. The angle between the axes of these reflections would be the sum of of the angles in the decompositions of the original rotations. If this angle is zero, and the lines are parallel, then the composition of rotations is a translation by Theorem *9*. If the angle is not zero, the axes intersect, then the composition of the rotations is a rotations around the intersection point by the angle which is the sum of angles of the original rotations.

Similar tricks with reflections allows to simplify other compositions.

Glide reflections. A reflection about a line l followed by a translation along l is called a *glide reflection*. In this definition, the order of reflection and translation does not matter, because they commute: $R_l \circ T_{AB} = T_{AB} \circ R_l$ if $l \parallel AB$.

Theorem 11. *The composition of a central symmetry and a reflection is a glide reflection.*

Use the same tricks as for Theorem *10*

Classification of plane isometries.

Theorem 12. *Any isometry of the plane is either a reflection about a line, a rotation, a translation, or a gliding reflection.*

This theorem can be deduced from Theorem *7* by taking into account relations between reflections in lines. By Theorem *7*, any isometry of the plane is a composition of at most 3 reflections about lines. By Theorems *8* and *9*, a composition of two reflections is either a rotation about a point or a translation.

Lemma. A composition of three reflections is either a reflection or a gliding reflection.

Proof. We will consider two cases: 1) all three lines are parallel, 2) not all lines are parallel (although two of the three may be parallel to one another).

The first one is easier; it is pretty straightforward to see (at least in some examples) that the composition is a translation. However, since the order of reflections matters, for a precise proof we wouold have to check different cases (if the lines are all vertical, the first reflection may be done about the leftmost, the rightmost, or the middle lien, etc.) To avoid this, we proceed as follows. Notice that $R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ (R_{l_2} \circ R_{l_1})$, and the composition $R_{l_2} \circ R_{l_1}$ of two reflections in parallel lines is a translation. This translation depends only on the direction of the lines and the distance between them, ie $R_{l_2} \circ R_{l_1} = R_{l'_2} \circ R_{l'_1}$ for any two lines l'_1, l'_2 that are parallel to l_1, l_2 and have the same distance between them. Thus, we translate the first two lines to make the second line coincide with the third, ie choose l'_1, l'_2 so that $l'_2 = l_3$. Then

$$R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ R_{l'_2} \circ R_{l'_1} = R_{l_3} \circ R_{l_3} \circ R_{l'_1} = R_{l'_1}$$

since two reflections about the same line l_3 cancel. Therefore, the result is a reflection (about the line l'_1).

If the three lines are not all parallel, then the second line l_2 is not parallel to l_1 or l_3. Let's suppose l_1 and l_2 are not parallel (the other case is very similar). Then the composition $R_{l_2} \circ R_{l_1}$ of reflections about intersecting lines is a rotation (that depends only on the point where the lines intersect, and the angle at which they intersect). So the lines l_1, l_2 can be rotated simultaneously about their intersection point by the same angle without changing the composition.

By an appropriate rotation, make the second line l_2 perpendicular to the third line l_3 (which is not rotated), ie replace l_1, l_2 by l'_1, l'_2 so that $R_{l_2} \circ R_{l_1} = R_{l'_2} \circ R_{l'_1}$, and $l'_2 \perp l_3$.

Then by rotating these two perpendicular lines l'_2, l_3 about their intersection point, make the middle line l_2 parallel to the line l_1. That is, we replace the lines l'_2, l_3 by lines l''_2, l''_3 so that

$$R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ R_{l'_2} \circ R_{l'_1} = R_{l''_3} \circ R_{l''_2} \circ R_{l'_1}.$$

Now, the configuration of lines consists of two parallel lines and a line perpendicular to them: l_1', l_2'' are parallel, l_3'' is perpendicular to them both. The composition of reflections $R_{l_2''} \circ R_{l_1'}$ is a translation by a vector perpendicular to these two lines (and thus parallel to the third); so $R_{l_3''} \circ (R_{l_2''} \circ R_{l_1'})$ is a glide symmetry. But the composition of these three reflections is the same as the composition of reflections about the original three lines.

Properties of the four types of isometries. We have just seen that any isometry of the plane belongs to one of the four types. How do we detect to which type it belongs? In particular, it may seem a bit mysterious that while composition of 3 reflections is a reflection or glide reflection, a composition of two isometries can never be a reflection, but only a rotation or translation. This can be explained as follows. Suppose our plane lies in the 3-space (as a horizontal xy-plane), and its top is painted black, its bottom white. Suppose that the reflections are done by rotating the plane around the line (axis of reflection) in the 3-space. Then after a reflection, the white side will be on top, the black side on the bottom. Notice that the colors will flip this way if we perform any odd number of reflections, but after an even number of reflections the colors do not flip. (Eg after two reflections, the top will be black again, the bottom white.) By contrast, rotations and translations do not flip the colors. This explains why the composition of two reflections can be a rotation or translation, but never a reflection.

Another fundamental characteristic of an isometry is the points that it leaves fixed. For instance, a rotation doesn't move the center (but moves any other point); a reflection fixes every point of its axis. We summarize these properties in the chart below.

type of isometry	points that stay fixed	flips colors?
rotation	the center	no
reflection	every point on axis	yes
translation	none	no
glide reflection	none	yes

These properties help detect the type of isometry. In particular, the chart shows that a glied reflection cannot belong to any of the other three types.

□