(1)

(a) Which of the following series converge?

(i) \( \sum_{n=1}^{\infty} \frac{1}{n^4 - n - 3} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^4} \). Indeed, for large \( n \) we have \( n^4 - n - 3 > \frac{1}{2} n^4 \), so \( \frac{1}{n^4 - n - 3} < \frac{2}{n^4} \). Because \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) converges, \( \sum_{n=1}^{\infty} \frac{1}{n^4 - n - 3} \) will also converge. (You would need more effort or a limit comparison argument to formalize this, but this reasoning is good enough to give you an answer. Note that all series here consist of positive terms.)

(ii) \( \sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1} \) diverges because the \( n \)-th term \( \frac{n^2}{n^2 + 1} \) converges to 1. (We know that \( \lim_{n \to \infty} a_n = 0 \) is a necessary condition for convergence of the series \( \sum_{n=1}^{\infty} a_n \).

(iii) \( \sum_{n=1}^{\infty} \frac{1}{3^n + n} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{3^n} \).

More specifically,

\[
3^n < 3^n + n \implies \frac{1}{3^n + n} < \frac{1}{3^n}.
\]

which converges. Since both series are made up of positive terms, this implies that the \( \sum_{n=1}^{\infty} \frac{1}{3^n + n} \) converges, because \( \sum_{n=1}^{\infty} \frac{1}{3^n} \) converges (the latter is a geometric series).

(iv) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} \) diverges. Indeed, we know that \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) diverges for \( p \geq 1 \).

We could also go by comparison with \( \sum_{n=1}^{\infty} \frac{1}{n} \). We have

\[
n \geq \sqrt[n]{n} \implies \frac{1}{\sqrt[n]{n}} \geq \frac{1}{n} \implies \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n}.
\]
Since the lower series diverges, both series diverge.

(b) Which of the following statements imply that \((x_n)\) is a Cauchy sequence?

(i) \(\lim_{n \to \infty} |x_n - x_{n+1}| = 0\)

This statement does not imply that \((x_n)\) is Cauchy. Otherwise the \(n\)-th term test for series would be enough to guarantee convergence.

For example, let \(x_n = \sum_{k=1}^{n} \frac{1}{k}\). Then \(|x_n - x_{n+1}| \leq |x_n| + |x_{n+1}| < 2/n \to 0\). However, we know \(x_n\) diverges.

Another example is provided by one of the practice questions: the sequence \((x_n)\) with \(x_n = \sqrt{n}\) satisfies the condition \(\lim_{n \to \infty} |x_n - x_{n+1}| = 0\), but the sequence is not Cauchy because it diverges to infinity.

(ii) For every \(\varepsilon > 0\), there exists an \(N \in \mathbb{N}\) such that for all \(m, p > N\) we have that \(|x_m - x_p| < \varepsilon\).

This is the definition of Cauchy sequence. Clearly, then, it implies that \((x_n)\) is Cauchy.

(iii) \(|x_n - x_{n+1}| = \frac{1}{2^n}\).

Since \(\sum_{n=1}^{\infty} \frac{1}{2^n} = 1\), for any \(\varepsilon > 0\), there must be some \(N \in \mathbb{N}\) for which

\[
\sum_{n=N}^{\infty} \frac{1}{2^n} < \varepsilon
\]  

Hence, \(m > n > N \implies \)

\[
|x_m - x_n| = \left| \sum_{k=n}^{m-1} (x_{k+1} - x_k) \right| \leq \sum_{k=n}^{m-1} |x_{k+1} - x_k| = \sum_{k=n}^{m-1} \frac{1}{2^k} \leq \sum_{k=N}^{\infty} \frac{1}{2^k} < \varepsilon
\]  

Thus \((x_n)\) is Cauchy.

(iv) If \((x_n)\) converges, then \((x_n)\) is Cauchy. In fact, we had a theorem saying that a sequence is Cachy if and only if it is convergent. (This is one of the corollaries of completeness of \(\mathbb{R}\)).

(c) Which of the following functions are uniformly continuous on the given intervals?
(i) \( f(x) = x^2 \) is uniformly continuous on \([0, 1]\) because it is continuous to begin with, and \([0, 1]\) is a closed bounded interval.

(ii) \( f(x) = x^2 \) is not uniformly continuous on \([0, \infty)\).

To show this, it is sufficient to find sequences \((x_n), (y_n)\) in \([0, \infty)\) such that

\[ |x_n - y_n| < 1/n \text{ but } |x_n^2 - y_n^2| > 1. \]

We can do this because the "slope" of \(x^2\) (in other words, the derivative) grows to infinity. Of course, we don't know about derivatives yet.

More formally, we can assume that \(x_n \leq y_n\) and write

\[ |x_n^2 - y_n^2| = y_n^2 - x_n^2 = (y_n + x_n)(y_n - x_n) \geq 2x_n(y_n - x_n). \]

Thus, let us take \(x_n = n\) and \(y_n = n + \frac{1}{2n}\). Then clearly \(|x_n - y_n| = \frac{1}{2n} < \frac{1}{n}\), while

\[ |x_n^2 - y_n^2| \geq 1. \]

Thus \(x^2\) cannot be uniformly continuous.

(iii) \( f(x) = \frac{1}{x^2+1} \) is uniformly continuous on \(\mathbb{R}\). This is because it is continuous, and its limits at \(\pm \infty\) are zero. A very similar question was discussed in Homework 9, Problem 4; see solutions for HW 9 for more details.

(iv) Consider the function defined on \([-1, 1]\) by \( f(0) = 0 \) and \( f(x) = x \sin(1/x) \) otherwise. This function is continuous; a very similar question with cosine instead of sine was on one of the homeworks. Intuitively, you can assume continuity from the graph. Since \( f(x) \) is continuous on a closed bounded interval, it must be uniformly continuous.

\(\blacksquare\)

(2) Let \((x_n)\) be a sequence in \([-1, 1]\). Then the sequence \(y_n = \max\{x_1, x_2, ..., x_n\}\) converges.

**Proof.** The sequence is increasing (because \(y_n = \max\{y_{n-1}, x_n\}\) and bounded, so it converges by the Monotone Convergence Theorem.

\(\blacksquare\)
(3) Let $f, g$ be two functions defined for all $x$, except perhaps at $x = 2$. Suppose $\lim_{x \to 2} f(x) = 0$ and $\lim_{x \to 2} g(x) = +\infty$. Then $\lim_{x \to 2} \frac{f(x)}{g(x)} = 0$.

**Proof.** Choose any $\varepsilon > 0$. Then for some $\delta_1 > 0$,

$$0 < |x - 2| < \delta_1 \implies g(x) > 1 \implies |g(x)| > 1$$

Next, for some $\delta_2 > 0$, we have

$$0 < |x - 2| < \delta_2 \implies |f(x)| < \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$0 < |x - 2| < \delta \implies \left| \frac{f(x)}{g(x)} - 0 \right| = \frac{|f(x)|}{|g(x)|} < \frac{\varepsilon}{1} < \varepsilon$$

Hence indeed $\lim_{x \to 2} \frac{f(x)}{g(x)}$ exists and is equal to 0.

✠

(4) Recall the

**Bolzano-Weierstrass Theorem.** Every bounded sequence has a convergent subsequence.

Let $f, g : [0, 1] \to \mathbb{R}$ be two continuous functions. Suppose that for some sequence $(x_n)$ in $[0, 1]$ we have $f(x_n) = g(x_n) + \frac{1}{n}$. Then $f(y) = g(y)$ for some $y \in [0, 1]$.

**Proof.** By Bolzano-Weierstrass, $(x_n)$ must have a convergent subsequence $x_{n_k}$. We will write its limit as $y$. Note that $0 \leq y \leq 1$ because $0 \leq x_{n_k} \leq 1$. By continuity,

$$f(y) = \lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} g(x_{n_k}) + \lim_{k \to \infty} \frac{1}{n_k} = g(y)$$

( Since $n_k \geq k$ goes to $\infty$, it follows that $\frac{1}{n_k}$ must go to 0.)

✠

(5) Let $f : [0, +\infty) \to (0, \infty)$ be a continuous function such that $\lim_{x \to \infty} f(x) = \infty$. 

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Then $f > M$ for some $M > 0$. (The statement may be false if we drop the requirement that $\lim_{x \to \infty} f(x) = \infty$.)

**Proof.** Since $\lim_{x \to \infty} f(x) = \infty$, we can find a number $R > 0$ such that

$$x > R \implies f(x) \geq 1 \quad (11)$$

Let $M' = \inf\{f(x) \mid x \in [0, R]\}$, and let $M = \frac{1}{2} \min\{1, M'\}$.

Then $M > 0$ if $M' > 0$. Indeed, $M' > 0$ because $f$ must attain its infimum $M'$ on the closed interval $[0, R]$, and $f$ is strictly positive. Thus, $f > M > 0$, as desired. ✠

For a counterexample in Part 2, any function with $\lim_{x \to \infty} f(x) = 0$ would do (it is then impossible to find $M > 0$ such that $f(x)$ remains greater than $M$ for all $x$).

✠