MAT 320 Introduction to Analysis

**Homework 8**
due Friday, November 10

1. Prove that the function $f(x) = 1/x$ is continuous at $x = 1$ by using the $\epsilon$-$\delta$-definition of a continuous function.

2. (a) Prove the Squeeze Theorem for functions: let $f, g, h : (a, b) \to \mathbb{R}$, $c \in (a, b)$. If $g(x) \leq f(x) \leq h(x)$ for all $x \in (a, b), x \neq c$, and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} f(x) = L$.

   (b) Use the Squeeze theorem to find $\lim_{x \to 0} x \cos(1/x)$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by
   
   $f(x) = \begin{cases} x \cos \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   is continuous at every $c \in \mathbb{R}$.

3. Consider a function $f : (a, b) \to \mathbb{R}$, $c \in (a, b)$.

   (a) Give a precise definition of the one-sided limits, $\lim_{x \to c}^+ f(x)$ and $\lim_{x \to c}^- f(x)$. State Sequential Criterion for existence of $\lim_{x \to c} f(x)$ (You don’t have to prove it.)

   (b) Prove that the limit of a function exists if and only if both one-sided limits exist and are equal, i.e. $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c}^+ f(x) = \lim_{x \to c}^- f(x) = L$.

4. Show that

   (a) For every $c \in \mathbb{R}$ there exists a sequence $(q_n)$ of rational numbers converging to $c$ (such that $q_n \neq c$ for all $n$).

   (b) For every $c \in \mathbb{R}$ there exists a sequence $(z_n)$ of irrational numbers converging to $c$ (such that $z_n \neq c$ for all $n$).

   We used this fact in class several times. Please give a careful proof. One way would be to use the Nested Intervals Property; another proof could be obtained via a careful application of the Archimedean Property.

5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions. Suppose that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x$. What can you say if you only know that $f(1/n) = g(1/n)$ for all $n \in \mathbb{N}$?

6. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that there exists $x \in [0, 1]$ such that $f(x) = x$. 