MAT 320 projects

The numbers in brackets indicate how many people can work on a given project. (For shorter projects, only two people are allowed to work together; for longer projects, up to four people is fine. Of course, if two people want to take up a longer project, they don't have to cover every aspect of it).

1. **Construction of reals.** (2–3) We developed a set of axioms for real numbers, but never discussed why an object satisfying these axioms actually exists. A "model" for real numbers can be obtained via *Dedekind cuts*. You could find out about those, and/or about some other interesting existence results: for instance, one can prove directly that $\sqrt{2}$ (i.e. a root of the equation $x^2 = 2$) exists in $\mathbb{R}$.
2. **Decimal presentations for rational and irrational numbers.**(2) There is a nice way to tell rational numbers from irrational by looking at their decimal presentation.
3. **Approximations by rationals.** (2–4) Every number can be approximated by rational numbers. Typically, you'll need to pick a rational number with a large denominator to get closer to the given number. For *Liouville numbers*, these approximations work really well: roughly, very close to a given number there are rational numbers whose denominators are not too large. It turns out that Liouville numbers are "transcendental"; you can establish existence of transcendentals, and even prove that e is a transcendental number.
4. **The Cantor set.** (2–3) This is a subset of $[0, 1]$ which can be defined in terms of ternary (base 3) presentation of real numbers. This set has a number of interesting properties, and is related to "fractals".
5. **Large and small sets.** (2–4)A subset of $\mathbb{R}$ can be large or small in a few different senses. For instance, we have finite and countable sets (small) and uncountable sets (large). There are also sets that are *dense*, i.e. contain points in any given small interval (large), or nowhere dense, i.e. contain gaps everywhere (small). Finally, we can talk about the *measure* (length) of a set, in particular, sets of "measure 0" (small) and sets of "full measure" (large). A set can be large in one sense and small in the other sense. You can find out about different definitions and the relation between them.
6. **Euler's number e.**(2–3) We gave one definition of e in class. There are a couple more, and you can find out why these definitions are equivalent. You can also prove that e is irrational. You could also find out how e is related to banking and compound interest.
7. **Convergence tests for series.** (2)In calculus, you might have seen a number of convergence tests for infinite series, such as the Root test and the Ratio test. Now you can find out why these tests actually work.
8. **Absolute convergence and conditional convergence.** (2) A convergent infinite series can converge *absolutely* or conditionally. (In

the first case, a series made up of absolute values of the terms of the original series converges; in the second case, it doesn't.) A beautiful theorem of Riemann says that you can "shuffle" terms of a conditionally convergent series to obtain any given number as the sum of the series!

9. **Monotone functions.** (2) These have some interesting properties. Theorem: a monotone function can be discontinuous only at countably many points.
10. **Compact sets.** (2-4) Many theorems we talked (or will talk) about (e.g. Nested Intervals Property, Bolzano-Weierstrass Theorem, convergence of Cauchy's sequences, Maximum theorem for continuous functions) were stated for a closed interval $[a, b]$. They are in fact true for all *compact* sets $X \subset \mathbb{R}$. You can learn what a compact set is, and why (some of) the theorems are true in this more general situation. (Compact sets can be generalized to several dimensions, and even to a much more general case of "topological spaces". This is why they are so important).
11. **Metric spaces.** (2-4) The notions of limit, continuous function, etc. can be defined not only for intervals (or other subsets of $\mathbb{R}$), but for some more general spaces (provided we have a "distance" between any two points of the space).
12. **Newton's method.** (2) This is a recursive procedure for finding roots of equations. (using derivatives, tangent lines, etc.) In particular, for the square root it gives the algorithm we discussed in class. You should find out why Newton's method works (at least in some cases), not only how it works.
13. **Uniform convergence of functions.** (2-4) This is an important topic we don't have time for in class. You could learn about the definition of uniform convergence (and how it's different from *pointwise* convergence), and prove an important theorem: if a sequence of continuous functions converges uniformly, the limit is a continuous functions. You could also mention applications to functions given as a sum of power series. Uniform convergence also allows to find derivatives or integrals of a limit of a sequence of functions.
14. **Approximations of functions.** (2-3) This project is related to uniform convergence: we want to approximate a given function by functions that are *uniformly* (i.e. at every point) close to a given one. It turns out that evry continuous function on $[a, b]$ can be approximated by polynomials (Stone- Weierstrass theorem).
15. **A continuous, nowhere differentiable function.** (2-3) It is easy (why?) to find an example of a function that is continuous, but not differentiable at a given point. A more interesting question is to find a function which is continuous on $[0, 1]$, but not differentiable at *any* point of the interval. You'll need uniform convergence to construct such a function.

16. Peano curve. (2–3) Sometimes theorems of mathematical analysis contradict our intuition. For example, there exists a Peano curve: a (continuous) curve that passes through every point of a square. In other words, there exists a continuous surjective function from an interval to a square. You'll need to make sense of ”continuous function to a square”, and use uniform convergence to construct a Peano curve.