MAT 319/320
Subsequences: Theorem 11.2 (ii), p.68.

Theorem 11.2(ii) asserts that if (s_n) is a sequence unbounded above, it has a subsequence s_{n_k} diverging to $+\infty$. Moreover, this subsequence can be chosen to be monotonic.

Proof. We'll build s_{n_k} so that $s_{n_k} > k$, for every k, and $s_{n_{k-1}} < s_{n_k}$, so that the subsequence is increasing. Indeed, we can find a term of the sequence that is greater than 1 (otherwise 1 would be an upper bound for (s_n)); pick this term to be s_{n_1}, so $s_{n_1} > 1$. Now, we would like to find the next term, s_{n_2} so that $s_{n_2} > 2$. This is possible because 2 is not an upper bound, but we have to be careful: because the terms of the subsequence must go in the same order as in (s_n), we also need to ensure that s_{n_2} comes after s_{n_1}, ie $n_2 > n_1$. **We didn't emphasize this point in class for lack of time.** So just saying that 2 is not an upper bound is not enough: maybe $s_{n_1} > 2$, but we cannot pick s_{n_1} the second time. So we set

$$M = \max(s_1, s_2, \ldots s_{n_1}, 2).$$

Now, since M is not an upper bound, there is a term s_n that is greater than M. This term has to come after s_{n_1} (ie $n > n_1$), because M is greater than all the terms we encounter up to s_{n_1}. If we pick this s_n for s_{n_2}, we'll get that

$$s_{n_2} > s_{n_1}, \qquad s_{n_2} > 2, \qquad n_2 > n_1.$$

Now we continue with this strategy to find s_{n_3} so that $s_{n_3} > 3$, $s_{n_3} > s_{n_2}$. Since 3 cannot be an upper bound (and neither can s_{n_2}), there is a term of the sequence that's larger than both, but as before we have to ensure that this term comes after s_{n_2}. So we set

$$M = \max(s_1, s_2, \ldots, s_{n_1}, s_{n_1+1}, \ldots, s_{n_2}, 3).$$

The sequence (s_n) is unbounded above, so M is not an upper bound, so there is some s_n such that $s_n > M$. This term necessarily comes after s_{n_2} (because M is larger than all preceding terms), and so we can choose s_{n_3} to be this s_n.

We continue this strategy to find (s_{n_k}). It is now easy to see (why?) that (s_{n_k}) increases and diverges to $+\infty$.

$\square$