MIDTERM I SOLUTIONS

1.

(a) We construct a polynomial that has a as a root. Since

$$\begin{aligned} a &= \sqrt{3+\sqrt[3]{5}} \\ a^2 &= 3+\sqrt[3]{5} \\ a^2-3 &= \sqrt[3]{5} \\ (a^2-3)^3 &= 5 \\ (a^2-3)^3-5 &= 0, \end{aligned}$$

the polynomial $p(x) = (x^2-3)^3 - 5$ has a as a root, so a is algebraic.

(b) The above polynomial expands as $p(x) = x^6 - 9x^4 + 27x^2 - 32$. Since the leading coefficient is 1, the rational roots theorem implies that any rational root of p must in fact be an integer. Now observe that

$$a = \sqrt{3+\sqrt[3]{5}} < \sqrt{3+\sqrt[3]{8}} = \sqrt{5} < 3,$$

and similarly

$$a = \sqrt{3+\sqrt[3]{5}} > \sqrt{3+\sqrt[3]{1}} = \sqrt{4} = 2.$$

There is no integer strictly between 2 and 3, so a cannot be rational.

2.

(a) The leading coefficients suggest that $s_n \to 5/4$. To prove this, let $\epsilon > 0$ be given. In order to find the required index N, we set $|5/4 - s_n| < \epsilon$. Note that we can immediately remove the absolute value, since $s_n < 5/4$ for all n. Then

$$\begin{aligned}
\frac{5}{4} - s_n &< \epsilon \\
\frac{5}{4} - \frac{5n-7}{4n+1} &< \epsilon \\
\frac{33}{16n+4} &< \epsilon \\
33 &< 16n\epsilon + 4\epsilon \\
33 - 4\epsilon &< 16n\epsilon \\
\frac{33}{16\epsilon} - \frac{1}{4} &< n.
\end{aligned}$$

It suffices to take $n > 33/16\epsilon$. (This inequality implies the last line in the above calculation.) Since the above steps are invertible, we conclude that $|5/4 - s_n| < \epsilon$ for all $n > 33/16\epsilon$. This proves that $s_n \to 5/4$.

(b) Recall that a sequence converges iff every subsequence converges to the same value. Thus, it will suffice to construct 2 subsequences that converge to different values. Observe that

$$t_{6n} = \cos(2\pi n) \equiv 1,$$

but

$$t_{6n+3} = \cos(2\pi(n+1)) \equiv -1.$$

Therefore we have $t_{6n} \to 1$ while $t_{6n+3} \to -1$, so t_n does not converge.

3.

(a) Let $M > 0$ be given. Since x_n converges to $+\infty$, there exists some index N_1 such that $x_n > M$ for all $n > N_1$. Similarly, since y_n converges to 7, there exists some index N_2 such that $|y_n - 7| < 1$ for all $n > N_2$. In particular, this implies that $y_n > 6$ for all $n > N_2$. Put $N = \max\{N_1, N_2\}$. Then for all $n > N$, we have

$$x_n + y_n > M + 6 > M.$$

Since M was arbitrary, we conclude that $x_n + y_n$ diverges to $+\infty$.

(b) Suppose for the sake of a contradiction that (x_{n_k}) is a decreasing subsequence. Since x_n diverges to $+\infty$, there can only be finitely many elements less than any given $M > 0$. (From the definition of divergence to $+\infty$, there exists some finite index N beyond which all terms in the sequence are larger than M.) Consider $M = x_{n_1} + 1$. Since (x_{n_k}) is decreasing, every element in this subsequence is less than M. This yields infinitely many elements of x_n that are less than M, a contradiction. We conclude that a sequence diverging to $+\infty$ cannot have a decreasing subsequence.

4. I claim that s_n is decreasing and bounded below. In particular,

$$\frac{3}{8} \leq s_{n+1} \leq s_n \leq 1$$

for all n. To establish this claim we use induction. Observe that $s_2 = 7/12$, and $3/8 \leq 7/12 \leq 1 \leq 1$. This proves the base case. Now assume the result is true up to n. Then

$$\begin{aligned} s_n &\geq \tfrac{3}{8} \\ \tfrac{2}{3} s_n &\geq \tfrac{1}{4} \\ s_n &\geq \tfrac{1}{4} + \tfrac{1}{3} s_n \\ s_n &\geq s_{n+1}. \end{aligned}$$

We also have

$$\begin{aligned} s_{n+1} &= \tfrac{1}{4} + \tfrac{1}{3} s_n \\ &\geq \tfrac{1}{4} + \tfrac{1}{3} \cdot \tfrac{3}{8} \\ &= \tfrac{3}{8}. \end{aligned}$$

This proves the claim. Since s_n is monotone and bounded, its limit exists. Call it s. We can now take the limit of both sides of $s_{n+1} = 1/4 + s_n/3$ to get $s = 1/4 + s/3$, which yields $s = 3/8$.

5.

(a) (i) Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

(ii) Every bounded sequence has a convergent subsequence.

(b) By the Archimedean property, there exists some positive integer N such that $1/N < \epsilon$. Therefore $1/N$ is rational and lies in the interval $(0, \epsilon)$.