MAT 319
More on Limits

There's a convenient reformulation of the standard definition of the limit of a sequence:

Theorem 1. *A sequence (x_n) converges to L if and only if, for any given $\epsilon > 0$, the inequality $|x_n - L| < \epsilon$ holds for* all but finitely many *values of n. (Equivalently, one can say that* all but finitely many *terms x_n are contained in the interval $(L - \epsilon, L + \epsilon)$).*

Proof. $\Rightarrow$ Suppose (x_n) converges to L. Given an arbitary $\epsilon > 0$, there is N such that $|x_n - L| < \epsilon$ for all $n > N$. (Equivalently, for all $n > N$, x_n is in the interval $(L - \epsilon, L + \epsilon)$.) Then, the only terms that can *possibly* lie outside of $(L - \epsilon, L + \epsilon)$ are those among $x_1, x_2, x_3, \ldots, x_N$; these form a finite set. The inequality $|x_n - L| < \epsilon$ can only fail for $n = 1, 2, \ldots N$, so it holds for *all terms except a finite number.*

Notice that some of the terms $x_1, x_2, \ldots x_N$ may happen to satisfy $|x_n - L| < \epsilon$ as well, so the finite collection of the "bad" terms might be even smaller.

$\Leftarrow$ Conversely, suppose that for any given $\epsilon > 0$, $|x_n - L| < \epsilon$ holds for all but finitely many terms. List all the "bad" terms $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$ for which the inequality fails. Choose $N = \max(n_1, n_2, \ldots n_k)$. (Observe that this is a finite set of natural numbers, so the maximum exists.)

This choice of N ensures that for any $n > N$, x_n cannot be a "bad" term from our finite collection (because $n > \max(n_1, n_2, \ldots n_k)$), so we must have $|x_n - L| < \epsilon$ whenever $n > N$. Since we were able to find an N for any given ϵ, the sequence (x_n) converges to L. $\square$

This theorem sometimes lets us get make shorter proofs; in particular, it allows to avoid relabeling things and estimating n_k's when working with subsequences. For example, we can make

A different proof for theorem 11.3 in the book. The theorem states that every subsequence of a convergent sequence converges to the same limit. Let's prove this.

Let (s_{n_k}) be a subsequence of (s_n). Suppose the sequence (s_n) converges to L. By the theorem above, this means that for any $\epsilon > 0$, *all but finitely many* terms s_n are contained in $(L - \epsilon, L + \epsilon)$. So only finitely many terms s_n can be outside of this interval. But the subsequence (s_{n_k}) is a selection of terms of (s_n), so there will be even fewer (or the same number, at most) of terms s_{n_k} outside of $(L - \epsilon, L + \epsilon)$. This means that all but finitely many s_{n_k} are in $(L - \epsilon, L + \epsilon)$ for every ϵ, so (s_{n_k}) converges to L. $\square$