HOMEWORK 9 SOLUTIONS

1. If $-f$ achieves its maximum at $x_0 \in [a,b]$, then by definition $-f(x_0) \geq -f(x)$ for all $x \in [a,b]$. Multiplying by -1 reverses the inequality, so this is equivalent to $f(x_0) \leq f(x)$ for all $x \in [a,b]$. Therefore f achieves its minimum at x_0.

Given a continuous function f, $-f$ is also continuous and therefore achieves its maximum on the closed interval $[a,b]$. The above argument shows that the maximum of $-f$ is the minimum of f, so f achieves its minimum on $[a,b]$.

2. Theorem 18.1 still shows that f is bounded, so $m = \inf\{f(x) : x \in [a,b]\}$ is a real number. Then for each $n \in \mathbb{N}$, $m + 1/n$ is not a lower bound for the range of f, so there exists some point x_n satisfying

$$m \leq f(x_n) < m + \frac{1}{n}.$$

By construction, $\lim f(x_n) = m$. Using the Bolzano-Weierstrass theorem, we can extract a convergent subsequence (x_{n_k}) with limit, say, $x_0 \in [a,b]$. Then by continuity of f,

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0).$$

But $f(x_n)$ is itself a convergent subsequence, so $\lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = m$. This shows $f(x_0) = m$, so we conclude that f achieves its minimum.

3. Since $f(x_0)$ is strictly less than c, the number $\epsilon = c - f(x_0)$ is positive. Then by definition of continuity, there exists some $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon$$

whenever $|x - x_0| < \delta$. This condition is equivalent to $x \in (x_0 - \delta, x_0 + \delta)$, and if $|f(x) - f(x_0)| < \epsilon$ then $f(x) < c$. This shows we have found the required δ.

4.

(a) Let $s = \lim s_n$ and choose $\epsilon = a - s$. Then by definition of convergence, there exists some N such that $|s_n - s| < \epsilon$ for all $n > N$. If $|s_n - s| < \epsilon = a - s$, then $s_n < a$. Therefore this N satisfies the desired property.

(b) Consider the contrapositive of the given statement: if $\lim s_n < a$, then $s_n < a$ for at least one n. We proved in part (a) that in fact, if $\lim s_n < a$, then $s_n < a$ for infinitely many n. Therefore part (a) implies the contrapositive of the given statement.

(c) Consider the sequence $t_n = -s_n$. Then $t_n \geq -b$ and (t_n) converges. Part (b) then implies that $\lim t_n \geq -b$, hence $\lim s_n = -\lim t_n \leq b$.

(d) No. Observe that $s_n = 1/n > 0$, but $\lim s_n = 0 \not> 0$. Similarly, $t_n = -1/n < 0$ but $\lim t_n = 0 \not< 0$.

18.4 Let $f(x) = 1/(x - x_0)$. Then

$$\lim |f(x_n)| = \lim \frac{1}{|x_n - x_0|}$$

and $|x_n - x_0| \to 0$ by assumption. This shows that $|f(x_n)| \to \infty$, so f is unbounded on S. f is continuous on S since the denominator $x - x_0$ is nonzero for all $x \in S$.

18.6 We apply the intermediate value theorem to $f(x) = x - \cos(x)$. Note that $f(0) = -1$ and $f(\pi/2) = \pi/2$. Since $0 \in (-1, \pi/2)$, the intermediate value theorem guarantees some $x_0 \in (0, \pi/2)$ such that $f(x_0) = 0$. Then $x_0 = \cos(x_0)$.