15.2

(a) The series diverges. Observe that for \( n = 6k + 3 \), \( \sin(n\pi/6)^n = 1 \). Since the terms of the series do not converge to zero, the sum cannot converge.

(b) The series converges. Since \( n/7 \) is never a half integer, \( \sin(n\pi/7) < 1 \) for all \( n \). In fact, \( \sin(n\pi/7) \) only takes on finitely many values, so we may choose some \( r \) such that \( 0 \leq |\sin(n\pi/7)| < r < 1 \) for all \( n \). Then

\[
\sum |\sin(n\pi/7)^n| = \sum |\sin(n\pi/7)|^n < \sum r^n < \infty,
\]

so the series of absolute values converges, i.e. the series converges absolutely. It is a fact (see corollary 14.7) that absolutely convergent series are convergent.

15.4

(a) The series diverges. Note that \( \log n < \sqrt{n} \) for all \( n \). (To see this, check that \( f(x) = \sqrt{x} - \log x \) is positive at 1 and always has positive derivative.) Then \( \frac{1}{\log n} > \frac{1}{\sqrt{n}} \), so \( \frac{1}{\sqrt{n} \log n} > \frac{1}{n} \). By comparison with the harmonic series, the given series diverges.

(b) The series diverges. Since \( \frac{\log n}{n} > \frac{1}{n} \), this series also diverges by comparison with the harmonic series. Alternatively, one can evaluate the integral

\[
\int_1^N \frac{\log x}{x} \, dx = \frac{\log^2 x}{2} \bigg|_1^N = \frac{\log^2 N}{2},
\]

which diverges to \( +\infty \) as \( N \to \infty \).

(c) The series diverges by the integral test:

\[
\int_4^N \frac{1}{x(\log x)(\log \log x)} \, dx = \log \log x \bigg|_4^N.
\]

\( \log \log \log N \) diverges to \( +\infty \) as \( N \to \infty \).
(d) The series converges. Again using $\log n < \sqrt{n}$, we have $rac{\log n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$. This is a $p$-series with $p = 3/2 > 1$, so it converges, and therefore the original series converges by comparison. Alternatively, one can evaluate the integral
\[
\int_{1}^{N} \frac{\log x}{x^2} \, dx = -\frac{1}{x} - \frac{\log x}{x} \bigg|_{1}^{N} = 1 - \frac{1}{N} - \frac{\log N}{N}.
\]
As $N \to \infty$ this converges to 1.

15.6

(e) The series $\sum (-1)^n \frac{1}{\sqrt{n}}$ converges by the alternating series test, but its squared terms form the harmonic series which diverges.