HOMEWORK 6 SOLUTIONS

14.4

(a) The series converges by the comparison test. Observe that $\frac{1}{[n+(-1)^n]^2} \leq \frac{1}{(n-1)^2}$ for all n. Using this comparison and then reindexing the sum, we get

$$\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2} \leq \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

(b) The series diverges by definition. The N-th partial sum is given by

$$\begin{aligned}\sum_{n=1}^{N} \sqrt{n+1} - \sqrt{n} &= (\sqrt{N+1} - \sqrt{N}) + (\sqrt{N} - \sqrt{N-1}) + \cdots + (2-1) \\ &= \sqrt{N+1} - 1,\end{aligned}$$

which becomes arbitrarily large.

(c) The series converges by the ratio test. The ratio of successive terms is given by

$$\begin{aligned}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} &= \left(\frac{n}{n+1}\right)^n \\ &= \left(\frac{n}{n+1}\right)^{n+1} \cdot \frac{n+1}{n} \\ &= \left(1 - \frac{1}{n+1}\right)^{n+1} \cdot \frac{n+1}{n}.\end{aligned}$$

Using the fact that $\lim(1-1/n)^n = 1/e$, we see that the limit of this ratio is $1/e < 1$.

15.6

(a) The series $\sum 1/n$ diverges, but $\sum (1/n)^2 = \sum 1/n^2$ converges.

(b) If $\sum a_n$ converges, the $a_n \to 0$. Then there exists some index N such that $a_n < 1$ for all $n > N$. Then we write

$$\begin{aligned}\sum a_n^2 &= \sum_{n=1}^{N} a_n^2 + \sum_{n=N+1}^{\infty} a_n^2 \\ &\le \sum_{n=1}^{N} a_n^2 + \sum_{n=N+1}^{\infty} a_n \\ &\le \sum_{n=1}^{N} a_n^2 + \sum_{n=1}^{\infty} a_n.\end{aligned}$$

The first sum contains finitely many terms, and therefore is finite, and the second sum is finite by assumption. This proves that $\sum a_n^2$ converges.

1.

(a) Choose some r such that $L < r < 1$. Then by definition of convergence, there exists some index N such that $\sqrt[n]{a_n} \le r$ for all $n > N$. (Apply the definition to $\epsilon = r - L$.) Then $a_n \le r^n$, so we have

$$\sum a_n \le \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} r^n.$$

The first sum contains finitely many terms and therefore converges, and the second sum is a geometric series with $r < 1$, which then also converges. This proves $\sum a_n$ converges.

(b) Choose some r such that $1 < r < L$. As before, take N such that $\sqrt[n]{a_n} \ge r$ for all $n > N$. Then $a_n \ge r^n$, so we have

$$\sum a_n \ge \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} r^n.$$

The second sum is a geometric series with $r > 1$, which diverges. This proves $\sum a_n$ diverges.

2.

(a) For all $\epsilon > 0$, there exists N such that $|a_n/b_n - L| < \epsilon$ for all $n > N$. Equivalently, $-\epsilon < a_n/b_n - L < \epsilon$, which after a bit of rearrangement yields

$$b_n(L-\epsilon) < a_n < b_n(L+\epsilon).$$

Then the comparison

$$\sum a_n \leq \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} b_n(L+\epsilon) = \sum_{n=1}^{N} a_n + (L+\epsilon) \sum_{n=N+1}^{\infty} b_n$$

shows that $\sum a_n$ converges.

(b) We'll use the other side of the inequality derived in part (a), $b_n(L-\epsilon) < a_n$. Choosing ϵ so that $L - \epsilon > 0$ (which is possible since $L > 0$), we can again use the comparison test to see that $\sum a_n$ diverges.

(c) The condition $L \neq 0$ was used in part (b).

3.

(a) Since the square of any number is nonnegative, $(\sqrt{a}-\sqrt{b})^2 \geq 0$. Expanding the square and moving the mixed term to the other side, we get $a + b \geq 2\sqrt{ab} \geq \sqrt{ab}$. This proves the hint. Using this,

$$\sum \sqrt{a_n b_n} \leq \sum a_n + b_n = \sum a_n + \sum b_n.$$

By the comparison test, $\sum \sqrt{a_n b_n}$ converges.

(b) Since $\sum a_n$ converges, $a_n \to 0$. Then there exists some N such that $a_n < 1$ for all $n > N$, so we have

$$\sum a_n b_n = \sum_{n=1}^{N} a_n b_n + \sum_{n=N+1}^{\infty} a_n b_n \leq \sum_{n=1}^{N} a_n b_n + \sum_{n=N+1}^{\infty} b_n.$$

The sum $\sum a_n b_n$ converges by the comparison test.

(c) No. Suppose the first two terms of each series are 1, and all the others are 0. Then $\sum a_n = \sum b_n = \sum a_n b_n = 2$, but $\sum a_n \cdot \sum b_n = 4$.