MAT 319 - Spring 2016 Homework 5 Solutions

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Question 1
(a) The sequence is given by 1, 1/2, 1, 1/2, 1/3, ... Although it is no use to write an explicit formula for \( s_n \) for general \( n \in \mathbb{N} \), one can notice that all of its terms have the form \( 1/m \) for \( m > 0 \). Clearly, in that case, the set \( \{1/n\}_{n \in \mathbb{N}} \cup \{0\} \) is included in the set of subsequential limits. To see this, for any integer \( n > 0 \), pick the subsequence \( (s_{n_k}) \) to be given by: \( s_{n_k} \in \{s_n\} \) for all \( n_k < N_k \) for a given fixed \( N_k \in \mathbb{N} \) and \( s_{n_k} = 1/m \) where \( m \) is any integer. Basically, you pick the first few (first \( N_k \)) elements to be any of the terms in the original sequence, and then pick the rest (the infinitely many terms after the \( N_k \)-th one) to be \( 1/m \) for various integers \( m \). As you can do that for any \( m \) (since any terms of the form \( 1/m \) for \( m > 0 \) is in \( (s_n) \)), that tells you that any \( 1/n, n \in \mathbb{N} \) will be a subsequential limit. For 0, simply pick the subsequence given by \( (s_{n_k}) = (1/k) \) for all \( k \in \mathbb{N} \). The claim is that these actually exhaust the set \( S \) of subsequential limits.

All that remains to show is that for any \( n \in \mathbb{N} \), there is a real number \( x \) such that \( x \neq 0 \) and \( x \neq 1/n \) can be a (subsequential) limit. Since \( 0 < s_n = 1/n \leq 1 \) in general, for any limit \( \ell \) we will have that \( 0 \leq \ell \leq 1 \) and so \( 0 < x \leq 1 \) (as we assume that \( x \) differs from 0). Now suppose that \( x \neq 1/n \) for any \( n \in \mathbb{N} \). Then we can pick two numbers of the form \( 1/n \) closest to \( x \). More precisely, \( x \) will be between \( 1/m \) and \( 1/(m+1) \) for appropriate \( m \). Hence, given \( \varepsilon < \min\{|a - 1/m|, |a - 1/(m+1)|\} \) for any \( m \), \( \varepsilon \) will be smaller than the distance from \( x \) to any of the terms of the sequence \( (s_n) \). Thus \( (x - \varepsilon, x + \varepsilon) \) will contain no terms of the sequence whatsoever, and so no subsequence can converge to \( x \). So \( S \) is indeed just \( \{1/n\}_{n \in \mathbb{N}} \cup \{0\} \).

Question 2
Let \( (x_n) \) and \( (y_n) \) be two sequences converging to the same number \( a \) and define \( (z_n) \) by \( z_{2n-1} = x_n \) and \( z_{2n} = y_n \) for all \( n \in \mathbb{N} \).

By the definition of the limit:
\[
\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}, \forall n > N_1 : |x_n - a| < \varepsilon
\]
\[
\forall \varepsilon > 0, \exists N_2 \in \mathbb{N}, \forall n > N_2 : |y_n - a| < \varepsilon
\]

Let \( \varepsilon > 0 \) and set \( N = \max\{2N_1 - 1, 2N_2\} \). Now let \( n > N \) be arbitrary. If \( n \) is odd, i.e. \( n = 2k - 1 \) for some positive integer \( k \), then \( k > N_1 \) since \( n > N \geq 2N_1 - 1 \) and so: \( |z_n - a| = |z_{2k-1} - a| = |x_k - a| < \varepsilon \). If \( n \) is even, i.e. \( n = 2k \) for some positive integer \( k \), then \( k > N_2 \) as \( n > N \geq 2N_2 \) and so: \( |z_n - a| = |z_{2k} - a| = |y_k - a| < \varepsilon \).

Therefore \( |z_n - a| < \varepsilon \) for all \( n > N \) and this completes the proof.
Alternative solution:

Since \((x_n)\) and \((y_n)\) both converge to \(a\), all but finitely many terms \(x_n\) and \(y_n\) satisfy \(|x_n - a| < \varepsilon\) and \(|y_n - a| < \varepsilon\) for any given \(\varepsilon > 0\). Now as \((z_n)\) is defined by \(z_{2n-1} = x_n\) and \(z_{2n} = y_n\), then the condition \(|z_n - a| < \varepsilon\) (for any given \(\varepsilon > 0\)) will either be equivalent to \(|x_n - a| < \varepsilon\) or \(|y_n - a| < \varepsilon\), depending on the parity of \(n\) and is therefore satisfied for all but finitely many terms \(z_n\) since that is the case for both \(x_n\) and \(y_n\) upon which the terms of \((z_n)\) are constructed. Therefore, the condition \(|z_n - a| < \varepsilon\) holds for all but finitely many terms \(z_n\) for any given \(\varepsilon > 0\) and thus \(\lim z_n = a\).

Question 3

Let \((s_n)\) and \((t_n)\) be two sequences such that they differ in only finitely many terms, let \(N_0 \in \mathbb{N}\) be the biggest index for which they differ so that \(s_n = t_n\) for all \(n > N_0\). Case (i): Assuming \((s_n)\) converges to \(L < \infty\), then by the definition of the limit:

\[
\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}, \forall n > N_1 : |s_n - L| < \varepsilon
\]

Now letting \(\varepsilon > 0\) and picking \(N = \max\{N_0, N_1\}\), then for any \(n > N, n > N_0\) as well, since \(N \geq N_0\) and so \(s_n = t_n\) for all \(n > N\). Additionally as \(n > N_1\) for the same reason (by construction), then it follows that \(|t_n - L| = |s_n - L| < \varepsilon\).

Therefore \((t_n)\) also converges to \(L\) by the definition of the limit.

(The basic trick here is to pick \(N\) sufficiently large so that both the inequality for the limit holds and \(s_n = t_n\) for any \(n > N\).)

Case (ii): Now if \((s_n)\) diverges to \(+\infty\), then for all \(M > 0\), there is an \(N_1 \in \mathbb{N}\) such that for all \(n > N_1\), \(s_n > M\) by definition. Again, just pick \(N\) to be the maximum of \(N_1\) and \(N_0\) and the result follows immediately. The last case in which \((s_n)\) diverges to \(-\infty\) is quasi-identical: simply be cautious to reckon that the definition states that for all \(M < 0\), there is an \(N_1 \in \mathbb{N}\) such that for all \(n > N_1, s_n < M\), and then again pick \(N\) to be the largest of \(N_0\) and \(N_1\).

Remark: The conditions \(|s_n - L| < \varepsilon\) and \(|t_n - L| < \varepsilon\), for any given \(\varepsilon > 0\), are equivalent for all but finitely many terms \(s_n\) and \(t_n\) since \(s_n = t_n\) for all but finitely many terms. Then the result follows again immediately. Note that similar equivalent conditions can be stated for the case in which \((s_n)\) diverges.

Question 4

A proof quite similar to that found in the notes posted online. We will construct a subsequence \((s_{n_k})\) so that \(s_{n_k} < -k\) while the sequence is strictly decreasing. Since \((s_n)\) is assumed to be unbounded below in part (iii) of Theorem 11.2, then we can find a term \(s_{n_1}\) of the sequence which is lesser than \(-1\); i.e. \(s_{n_1} < -1\). Now let \(M_2 = \min\{s_1, s_2, \ldots, s_{n_1}, -2\}\). As \(M_2\) is not an lower bound, there is a term \(s_{n_2}\) of \((s_n)\) which is lesser than \(M_2\). Therefore, \(s_{n_2} < s_{n_1}, s_{n_2} < -2\) and \(n_2 > n_1\) since \(s_{n_2}\) is none of the terms encountered before (including \(s_{n_1}\)). Proceeding in a similar fashion, we can always set \(M_k\) to be \(\min\{s_1, \ldots, s_{n_1}, s_{n_1+1}, \ldots, s_{n_2}, s_{n_2+1}, \ldots, s_{n_{k-1}}, -k\}\) where \(s_{n_{k-1}}\) is constructed in the same fashion as \(s_{n_2}\), and again pick \(s_{n_k}\) to be a term less than \(M_k\) not encountered before. This sets up an inductive process insuring that \(n_k > n_{k-1}\), \(s_{n_{k-1}} < s_{n_k}\) and \(s_{n_k} < -k\) for all \(k\). Therefore \((s_{n_k})\) is strictly decreasing and unbounded below, thus diverging to \(\infty\).
Question 5

(a) $\sum \frac{n^2}{n^3} = \sum \frac{1}{n} + \sum \frac{1}{n^2}$. Now $\sum \frac{1}{n}$ diverges while $\sum \frac{1}{n^2}$ converges and so their sum as a series diverges.

(b) $\sum (-1)^n$. Assume it converges. Then $\lim(-1)^n = 0$ by Corollary 14.5. Clearly not the case, and so the series diverges.

(c) $\sum 3n/n^3 = \sum 3/n^2 = 3 \sum 1/n^2$ and $\sum 1/n^2$ converges. So our given series converges.

(d) $\sum \frac{n^3}{n!}$. Ratio Test: let $a_n = n^3/3^n$ and consider $a_{n+1}/a_n = (n+1)^3/3^{n+1}$ whose limit is $1/3$. So the series converges.

(e) $\sum n^2/n!$. Again, Ratio Test: let $a_n = n^2/n!$ and consider $a_{n+1}/a_n = (n+1)^2/n^2$. Its limit is $0 < 1$ and so the series converges.

(f) $\sum \frac{1}{n^{3/2}}$. Root Test: let $a_n = \frac{1}{n^{3/2}}$. Then: $|a_n|^{1/n} = 1/n$ whose limit is 0. Therefore the series converges absolutely.

(g) $\sum \frac{1}{n^{3/4}}$. Ratio Test: let $a_n = \frac{1}{n^{3/4}}$. Then: $a_{n+1}/a_n = 1/2 + 1/2n$ whose limit is $1/2 < 1$. Therefore the series converges.

Question 6

Let $a_n, b_n > 0$ for all $n$. Assume that $\sum a_n$ converges and that the sequence $(b_n)$ is bounded above. Since $(b_n)$ is bounded above, there exists an $M > 0$ such that $|b_n| < M$. Therefore, for all $n$ we have:

\[ a_n b_n \leq a_n |b_n| \leq M a_n \]

Now since $a_n > 0$ and $\sum a_n$ converges, then $\sum M a_n$ converges as well, and so $\sum a_n b_n$ must converge by the Comparison Test.