MAT 319 - Spring 2016 Homework 5 Solutions

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Question 1

(a) The sequence is given by $1, 1/2, 1, 1, 1/2, 1/3, ...$ Although it is no use to write an explicit formula for s_n for general $n \in \mathbb{N}$, one can notice that all of its terms have the form $1/m$ for $m > 0$. Clearly, in that case, the set $\{1/n\}_{n\in\mathbb{N}} \cup \{0\}$ is included in the set of subsequential limits. To see this, for any integer $n > 0$, pick the subsequence (s_{n_k}) to be given by: $s_{n_k} \in \{s_n\}$ for all $n_k < N_k$ for a given fixed $N_k \in \mathbb{N}$ and $s_{n_k} = 1/m$ where m is any integer. Basically, you pick the first few (first N_k) elements to be any of the terms in the original sequence, and then pick the rest (the infinitely many terms after the N_k-th one) to be $1/m$ for various integers m. As you can do that for any m (since any terms of the form $1/m$ for $m > 0$ is in (s_n)), that tells you that any $1/n, n \in \mathbb{N}$ will be a subsequential limit. For 0, simply pick the subsequence given by $(s_{n_k}) = (1/k)$ for all $k \in \mathbb{N}$. The claim is that these actually exhaust the set S of subsequential limits. All that remains to show is that for any $n \in \mathbb{N}$, there is no real number x such that $x \neq 0$ and $x \neq 1/n$ can be a (subsequential) limit. Since $0 < s_n = 1/n \leq 1$ in general, for any limit ℓ we will have that $0 \leq \ell \leq 1$ and so $0 < x \leq 1$ (as we assume that x differs from 0). Now suppose that $x \neq 1/n$ for any $n \in \mathbb{N}$. Then we can pick two numbers of the form $1/n$ closest to x. More precisely, x will be between $1/m$ and $1/(m+1)$ for appropriate m. Hence, given $\varepsilon < \min\{|a - 1/m|, |a - 1/(m+1)|\}$ for this m, ε will be smaller than the distance from x to any of the terms of the sequence (s_n). Thus $(x - \varepsilon, x + \varepsilon)$ will contain no terms of the sequence whatsoever, and so no subsequence can converge to x. So S is indeed just $\{1/n\}_{n\in\mathbb{N}} \cup \{0\}$.

Question 2

Let (x_n) and (y_n) be two sequences converging to the same number a and define (z_n) by $z_{2n-1} = x_n$ and $z_{2n} = y_n$ for all $n \in \mathbb{N}$.
By the definition of the limit:

$$\forall\varepsilon > 0, \exists N_1 \in \mathbb{N}, \forall n > N_1 : |x_n - a| < \varepsilon$$

$$\forall\varepsilon > 0, \exists N_2\mathbb{N}, \forall n > N_2 : |y_n - a| < \varepsilon$$

Let $\varepsilon > 0$ and set $N = \max\{2N_1 - 1, 2N_2\}$. Now let $n > N$ be arbitrary. If n is odd, i.e. $n = 2k - 1$ for some positive integer k, then $k > N_1$ since $n > N \geq 2N_1 - 1$ and so: $|z_n - a| = |z_{2k-1} - a| = |x_k - a| < \varepsilon$. If n is even, i.e. $n = 2k$ for some positive integer k, then $k > N_2$ as $n > N \geq 2N_2$ and so: $|z_n - a| = |z_{2k} - a| = |y_k - a| < \varepsilon$.
Therefore $|z_n - a| < \varepsilon$ for all $n > N$ and this completes the proof.

Alternative solution:
Since (x_n) and (y_n) both converge to a, all but finitely many terms x_n and y_n satisfy $|x_n - a| < \varepsilon$ and $|y_n - a| < \varepsilon$ for any given $\varepsilon > 0$. Now as (z_n) is defined by $z_{2n-1} = x_n$ and $z_{2n} = y_n$, then the condition $|z_n - a| < \varepsilon$ (for any given $\varepsilon > 0$) will either be equivalent to $|x_n - a| < \varepsilon$ or $|y_n - a| < \varepsilon$, depending on the parity of n and is therefore satisfied for all but finitely many terms z_n since that is the case for both x_n and y_n upon which the terms of (z_n) are constructed. Therefore, the condition $|z_n - a| < \varepsilon$ holds for all but finitely many terms z_n for any given $\varepsilon > 0$ and thus $\lim z_n = a$.

Question 3
Let (s_n) and (t_n) be two sequences such that they differ in only finitely many terms, let $N_0 \in \mathbb{N}$ be the biggest index for which they differ so that $s_n = t_n$ for all $n > N_0$. Case (i): Assuming (s_n) converges to $L < \infty$, then by the definition of the limit:

$$\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}, \forall n > N_1 : |s_n - L| < \varepsilon$$

Now letting $\varepsilon > 0$ and picking $N = \max\{N_0, N_1\}$, then for any $n > N$, $n > N_0$ as well, since $N \geq N_0$ and so $s_n = t_n$ for all $n > N$. Additionally as $n > N_1$ for the same reason (by construction), then it follows that $|t_n - L| = |s_n - L| < \varepsilon$. Therefore (t_n) also converges to L by the definition of the limit.
(The basic trick here is to pick N sufficiently large so that both the inequality for the limit holds and $s_n = t_n$ for any $n > N$.)
Case (ii): Now if (s_n) diverges to $+\infty$, then for all $M > 0$, there is an $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $s_n > M$ – by definition. Again, just pick N to be the maximum of N_1 and N_0 and the result follows immediately. The last case in which (s_n) diverges to $-\infty$ is quasi-identical: simply be cautious to reckon that the definition states that for all $M < 0$, there is an $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $s_n < M$, and then again pick N to be the largest of N_0 and N_1.

<u>Remark:</u> The conditions $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, for any given $\varepsilon > 0$, are equivalent for all but finitely many terms s_n and t_n since $s_n = t_n$ for all but finitely many terms. Then the result follows again immediately. Note that similar equivalent conditions can be stated for the case in which (s_n) diverges.

Question 4
A proof quite similar to that found in the notes posted online. We will construct a subsequence (s_{n_k}) so that $s_{n_k} < -k$ while the sequence is strictly decreasing. Since (s_n) is assumed to be unbounded below in part (iii) of Theorem 11.2, then we can find a term s_{n_1} of the sequence which is lesser than -1; i.e. $s_{n_1} < -1$. Now let $M_2 = \min\{s_1, s_2, ..., s_{n_1}, -2\}$. As M_2 is not an lower bound, there is a term s_{n_2} of (s_n) which is lesser than M_2. Therefore, $s_{n_2} < s_{n_1}$, $s_{n_2} < -2$ and $n_2 > n_1$ since s_{n_2} is none of the terms encountered before (including s_{n_1}). Proceeding in a similar fashion, we can always set M_k to be $\min\{s_1, ..., s_{n_1}, s_{n_1+1}, ..., s_{n_2}, s_{n_2+1}, ..., s_{n_{k-1}}, -k\}$ where $s_{n_{k-1}}$ is constructed in the same fashion as s_{n_2}, and again pick s_{n_k} to be a term less than M_k not encountered before. This sets up an inductive process insuring that $n_k > n_{k-1}$, $s_{n_{k-1}} < s_{n_k}$ and $s_{n_k} < -k$ for all k. Therefore (s_{n_k}) is strictly decreasing and unbounded below, thus diverging to ∞.

Question 5
(a) $\sum \frac{n-1}{n^2} = \sum \frac{1}{n} + \sum \frac{-1}{n^2}$. Now $\sum \frac{1}{n}$ diverges while $\sum \frac{-1}{n^2}$ converges and so their sum as a series diverges.
(b) $\sum(-1)^n$. Assume it converges. Then $\lim(-1)^n = 0$ by Corollary 14.5. Clearly not the case, and so the series diverges.
(c) $\sum 3n/n^3 = \sum 3/n^2 = 3\sum 1/n^2$ and $\sum 1/n^2$ converges. So our given series converges.
(d) $\sum \frac{n^3}{3^n}$. Ratio Test: let $a_n = n^3/3^n$ and consider $a_{n+1}/a_n = (n+1)^3/3n^3$ whose limit is $1/3$. So the series converges.
(e) $\sum n^2/n!$. Again, Ratio Test: let $a_n = n^2/n!$ and consider $a_{n+1}/a_n = (n+1)/n^2$. Its limit is $0 < 1$ and so the series converges.
(f) $\sum \frac{1}{n^n}$. Root Test: let $a_n = \frac{1}{n^n}$. Then: $|a_n|^{1/n} = 1/n$ whose limit is 0. Therefore the series converges absolutely.
(g) $\sum \frac{n}{2^n}$. Ratio Test: let $a_n = \frac{n}{2^n}$. Then: $a_{n+1}/a_n = 1/2 + 1/2n$ whose limit is $1/2 < 1$. Therefore the series converges.

Question 6
Let $a_n, b_n > 0$ for all n. Assume that $\sum a_n$ converges and that the sequence (b_n) is bounded above. Since (b_n) is bounded above, there exists an $M > 0$ such that $|b_n| < M$. Therefore, for all n we have:

$$a_n b_n \leq a_n|b_n| \leq Ma_n$$

Now since $a_n > 0$ and $\sum a_n$ converges, then $\sum Ma_n$ converges as well, and so $\sum a_n b_n$ must converge by the Comparison Test.