4.2

(a) $-2, -1, 0$.

(b) $-2, -1, 0$.

(c) $0, 1, 2$.

(d) $0, 1, 2$.

(e) $-2, -1, 0$.

(f) $-3, -2, -1$.

(g) $-2, -1, 0$.

(h) $0, 1, 2$.

(i) $-10, -2, -1$.

(j) $0, 1/3, 2/3$.

(k) $-10, -5, 0$.

(l) Not bounded below.

(m) $-10, -5, -2$.

(n) $-4, -3, -2$.

(o) Not bounded below.

(p) $-1, 0, 1$.

(q) $-4, -2, -1$.

(r) $-1, 0, 1$.

(s) $-2, -1, 0$.

(t) Not bounded below.

(u) Not bounded below.

(v) $-3, -2, -1$.

(w) $-3, -2, -1$.

4.6

(a) Let $a \in S$ be an arbitrary element. Then $\inf S \leq a \leq \sup S$.

(b) Let $B = \inf S = \sup S$. Then for every $a \in S$, we have $B \leq a \leq B$, so $a = B$. We conclude that S contains a single element.

4.10 It is known that there exist positive integers n_1, n_2 such that $1/n_1 < a$ and $a < n_2$. Put $n = \max\{n_1, n_2\}$. Then

$$\frac{1}{n} \leq \frac{1}{n_1} < a < n_2 \leq n.$$

4.12 By the denseness of $\mathbb{Q}$, there exists a rational number r such that $a - \sqrt{2} < r < b - \sqrt{2}$. Then $a < r + \sqrt{2} < b$, so all that remains to be shown is that $r + \sqrt{2}$ is irrational. Suppose for the sake of a contradiction that $x = r + \sqrt{2}$ is rational. Then since the rationals are closed under addition/subtraction, $x - r = \sqrt{2}$ must be rational, a contradiction. Therefore $r + \sqrt{2}$ is irrational.

4.14

(a) Observe that for any $a \in A$, we have

$$a = (a + b) - b \leq \sup(A + B) - b.$$

It follows that $\sup A \leq \sup(A + B) - b$. But then $b \leq \sup(A + B) - \sup A$ for arbitrary $b \in B$, so $\sup B \leq \sup(A + B) - \sup A$. We have therefore shown that $\sup A + \sup B \leq \sup(A + B)$.

To prove the reverse inequality, observe that for any $a + b \in A + B$, we have $a + b \leq \sup A + \sup B$. Therefore $\sup(A + B) \leq \sup A + \sup B$. We conclude that $\sup(A + B) = \sup A + \sup B$.

(b) The proof is identical to that of part (a).

5.6 If $\inf T = -\infty$, then $\inf T \leq \inf S$ is trivially satisfied. Therefore we may assume that T has a finite lower bound. For any $a \in S$, we know that $a \in T$, so $\inf T \leq a$. This means that $\inf T$ is a lower bound for S, so $\inf T \leq \inf S$. Similarly, if $\sup T = \infty$ then $\sup S \leq \sup T$ is trivial, so assume that $\sup T$ is finite. Then $\sup T \geq a$, so $\sup T$ is an upper bound for S, hence $\sup S \leq \sup T$. Putting the two inequalities together yields the desired result.

7.2

(a) $s_n \to 0$.

(b) $b_n \to 3/4$.

(c) $c_n \to 0$.

(d) Does not converge.