MAT 319 - Spring 2016 Homework 1 Solutions

TA: El Mehdi Ainasse - R01

February 5, 2016

Section 1

1.2 We want to show that $3+11+\ldots+(8n-5) = 4n^2 - n$ for all $n > 0$. Let P_n be that proposition. Clearly, we simply apply the principle of mathematical induction. You can check that the identity holds true for $n = 1$ as $3 = 4\times 1^2 - 1$. So P_1 is true. Now for the induction step, suppose that P_n is true for a fixed $n \geq 1$ and let's prove that P_{n+1} also holds true. We have the following:

$$3+11+...+(8(n+1)-5) = 3+11+\ldots+(8n-5)+(8(n+1)-5) = (4n^2-n)+(8n+8-5)$$

The passage from the second step to the third step following, of course, from our induction hypothesis. Now note that $4n^2-n+8n+8-5 = 4n^2+8n+4-n-1 = 4(n^2+2n+1)-(n+1) = 4(n+1)^2-(n+1)$, which shows that P_{n+1} is also true and the proof is thus complete by the principle of mathematical induction.

1.4 (a) For $n = 2$, it's $4 = 2^2$. For $n = 3$, it's $9 = 3^2$. For $n = 4$, it's $16 = 4^2$. My guess would be that the sum is n^2 in general for a given n.

(b) Let's prove the claim in (a) using the principle of mathematical induction. Let P_n be the proposition "$3 + 11 + \ldots + (2n-1) = n^2$ for all $n \in \mathbb{N}$". As the basis is already established (since P_1 is true), let us readily assume that P_n holds true for a fixed $n \geq 1$ and show that P_{n+1} follows:

$$3+11+\ldots+(2(n+1)-1) = 3+11+\ldots+(2n-1)+(2(n+1)-1) = n^2+(2n+2-1)$$

As the last expression is simply $n^2 + 2n + 1 = (n+1)^2$, we are done.

1.7 Let P_n be the proposition that $7^n - 6n - 1$ is divisible by 36 for all positive n. We will prove the given proposition by induction. P_1 is clearly true since $7^1 - 6\times 1 - 1 = 0$ is trivially divisible by 36. Now assume P_n holds for a fixed $n \geq 1$. Then:

$$7^{n+1} - 6(n+1) - 1 = 7\times 7^n - 42n - 7 + 36n = 7\times(7^n - 6n - 1) + 36n$$

Now as $7^n - 6n - 1$ is divisible by 36 by our induction hypothesis and is $36n$ is clearly divisible by 36, P_{n+1} holds true, and this completes the proof.

1.10 Induction, as usual. Note that the formula may be rewritten as follows:

$$(2n+1) + (2n+3) + (2n+5) + \ldots + (2n + (2n-1)) = 3n^2$$

Check that the formula holds for $n = 1$. Now call the proposition P_n and prove P_{n+1} under the assumption that P_n is true for all $n \geq 1$:

$(2(n+1)+1)+(2(n+1)+3)+(2(n+1)+5)+\ldots+(2(n+1)+(2(n+1)-1))$
$= (2n+3)+(2n+5)+\ldots+(2(n+1)+(2(n+1)-5))+(2(n+1)+(2(n+1)-3))+(2(n+1)+(2(n+1)-1))$
$= ((2n+1)+(2n+3)+(2n+5)+\ldots+(2n+(2n-1)))-(2n+1)+(4n+1)+(4n+3)$
$= 3n^2+6n+3 = 3(n+1)^2$, hence P_{n+1} holds true, and this completes the proof.

Section 2

2.2 The given numbers are roots of the polynomials x^3-2, x^7-5 and x^4-13. By the Rational Zeros Theorem, the only possible rational roots are: $\pm1, \pm2$ for x^3-2; $\pm1, \pm5$ for x^7-5; and $\pm1, \pm13$ for x^4-13.
Clearly, none of these are roots for the given polynomials, respectively. (Note that you should still check that.) Hence, none of the given numbers are rational.

2.4 Let $a = \sqrt[3]{5-\sqrt{3}}$. Then $a^3 = 5-\sqrt{3}$, and so $(5-a^3)^2-3 = 0$, i.e. $a^6-10a^3+22 = 0$. Now if the polynomial x^6-10x^3+22 has a rational root, it must be one of the following numbers: $\pm1, \pm2, \pm11, \pm22$ by the Rational Zeros Theorem. You can easily check that none of these are in fact roots of that polynomial. (Note that you should still check that for all candidates.) For example, 1 wouldn't work because when you evaluate the polynomial at 1 you obtain $1^6-10\times1^3+22 = 13$.
An alternative approach would be to notice that if we assume that $a \in \mathbb{Q}$, then $a^3 \in \mathbb{Q}$ as well so that $b = 5-a^3 \in \mathbb{Q}$ and thus b is a rational root of x^2-3, which is impossible by the Rational Zeros Theorem since the only possible roots would be $\pm1, \pm3$ and none of them is a root of x^2-3.

Section 3

3.4 (v) By (iv) from Theorem 3.2, $0 \leq a^2$ for all $a \in \mathbb{R}$. Hence, $0 \leq 1^2 = 1$. Now since $1 \neq 0$, it follows that $0 < 1$. (Note that to show that $1 \neq 0$, it suffices to argue by contradiction: for any non-zero a, $0 \cdot a = 0$ while $1 \cdot a = a$ and so if $1 = 0$, then: $0 \neq a = 1 \cdot a = 0 \cdot a = 0$, which is clearly absurd.)

(vii) Let $a, b \in \mathbb{R}$ and suppose that $0 < a < b$. Clearly, $0 < a$ and $0 < b$ and so $0 < a^{-1}$ and $0 < b^{-1}$ by (vi) in Theorem 3.2. Now by O5, given that $0 < a$ and $a < b$, it follows that $aa^{-1} < ba^{-1}$; i.e. $1 < ba^{-1}$. Similarly, we have $b^{-1} < b^{-1}ba^{-1} = a^{-1}$; i.e. $b^{-1} < a^{-1}$, and since $0 < b^{-1}$, the result follows.

3.6 (a) Let $a, b, c \in \mathbb{R}$, then by a double application of the triangle inequality:

$$|a+b+c| = |(a+b)+c| \leq |a+b|+|c| \leq |a|+|b|+|c|$$

(b) Establishing the basis case has already been done before (cf. textbook, for example). Now suppose that $|a_1+\ldots+a_n| \leq |a_1|+\ldots+|a_n|$ for n numbers $a_1, \ldots, a_n$. Let a_{n+1} be another number. Then:

$$|a_1+\ldots+a_{n+1}| = |(a_1+\ldots+a_n)+a_{n+1}| \leq |a_1+\ldots+a_n|+|a_{n+1}| \leq |a_1|+\ldots+|a_n|+|a_{n+1}|$$

Hence the result by the principle of mathematical induction.