29.2 Consider \( \frac{\cos x - \cos y}{x - y} \).

By the mean value theorem, there exists some \( c \in (a, b) \) with \( \cos'(c) \) equal to the above expression. But \( \cos'(c) = -\sin(c) \), which has absolute value at most 1. Therefore the above expression has absolute value at least 1, which proves the desired result.

29.7 (a) If \( f''(x) = 0 \) for all \( x \in I \), then by corollary 29.4, \( f'(x) = a \) for some constant \( a \). Now consider \( g(x) = f(x) - ax \). Since \( g'(x) = f'(x) - a = 0 \) for all \( x \in I \), \( g(x) = b \) for some constant \( b \). Therefore \( b = f(x) - ax \), i.e. \( f(x) = ax + b \).

(b) This is the same argument as above. If \( f'''(x) = 0 \) for all \( x \in I \), then \( f''(x) = a \) for some constant \( a \). Consider \( g(x) = f'(x) - ax \). Then \( g'(x) = f''(x) - a = 0 \), so \( g(x) = b \) for some constant \( b \). This shows \( f'(x) - ax - b = 0 \). Now set \( h(x) = f(x) - ax^2/2 - bx \). Since \( h'(x) = f'(x) - ax - b = 0 \), we have \( h(x) = c \) for some constant \( c \). This shows \( f(x) = ax^2/2 + bx + c \).

29.8 The set-up for all 3 remaining parts is the same. Consider any \( x_1, x_2 \) with \( a < x_1 < x_2 < b \). By the mean value theorem, there exists some \( x \in (x_1, x_2) \) such that

\[
f'(x) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.
\]

(ii) \( f'(x) < 0 \) and \( x_1 < x_2 \) imply that \( f(x_1) > f(x_2) \), so \( f \) is strictly decreasing.

(iii) \( f'(x) \geq 0 \) and \( x_1 < x_2 \) imply that \( f(x_1) \leq f(x_2) \), so \( f \) is increasing.

(iv) \( f'(x) \leq 0 \) and \( x_1 < x_2 \) imply that \( f(x_1) \geq f(x_2) \), so \( f \) is decreasing.
29.10

(a)

\[ f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x) + x/2}{x} = \lim_{x \to 0} x \sin(1/x) + 1/2. \]

Since the limits of both terms exist, and in particular \( \lim_{x \to 0} x \sin(1/x) = 0 \) and \( \lim_{x \to 0} 1/2 = 1/2 \), we can use the addition law to get that \( f'(0) = 1/2 > 0 \).

(b) By corollary 29.7 and assignment 11, a function \( f \) is increasing on some interval if and only if \( f' \geq 0 \) on that interval. Thus, it will suffice to show that in any open interval containing 0, there exists some \( x \) with \( f'(x) < 0 \). First we compute the derivative using the product and chain rules.

\[ f'(x) = 2x \sin(1/x) - \cos(1/x) + 1/2. \]

Now consider \( y_n = 1/2 \pi n, \ n \in \mathbb{N} \). Note that \( f'(y_n) = -1/2 \) for all \( n \), and that any open interval containing 0 must contain some \( y_n \). This proves that \( f \) cannot be increasing on any open interval containing 0.

(c) Although \( f'(0) > 0 \), the derivative is not positive (or even nonnegative) in any neighborhood of 0 so 29.7 does not apply. This is because the derivative is not continuous at 0.

29.13 Consider the auxiliary function \( h(x) = f(x) - g(x) \). Then \( h(0) = 0 \) and \( h'(x) \leq 0 \) for all \( x \in \mathbb{R} \). By corollary 29.7 \( h \) is decreasing on \( \mathbb{R} \). In particular, for any \( x \geq 0 \) we have \( h(x) \leq h(0) = 0 \). Then \( f(x) - g(x) \leq 0 \), so \( f(x) \leq g(x) \) for all \( x \geq 0 \).

29.14 This is an application of the previous problem. Let \( f_1(x) = x \), so \( f_1(0) = 0 = f(0) \) and \( f'_1(x) = 1 \leq f'(x) \). By the previous problem, \( x \leq f(x) \) for all \( x \geq 0 \). Now let \( f_2(x) = 2x \), so \( f_2(0) = 0 = f(0) \) and \( f(x) \leq 2 = f'_2(x) \). By the previous problem, \( f(x) \leq 2x \) for all \( x \geq 0 \).