HOMEWORK 12 SOLUTIONS

29.2 Consider

$$\frac{\cos x - \cos y}{x - y}.$$

By the mean value theorem, there exists some $c \in (a, b)$ with $\cos'(c)$ equal to the above expression. But $\cos'(c) = -\sin(c)$, which has absolute value at most 1. Therefore the above expression has absolute value at least 1, which proves the desired result.

29.7

(a) If $f''(x) = 0$ for all $x \in I$, then by corollary 29.4, $f'(x) = a$ for some constant a. Now consider $g(x) = f(x) - ax$. Since $g'(x) = f'(x) - a = 0$ for all $x \in I$, $g(x) = b$ for some constant b. Therefore $b = f(x) - ax$, i.e. $f(x) = ax + b$.

(b) This is the same argument as above. If $f'''(x) = 0$ for all $x \in I$, then $f''(x) = a$ for some constant a. Consider $g(x) = f'(x) - ax$. Then $g'(x) = f''(x) - a = 0$, so $g(x) = b$ for some constant b. This shows $f'(x) - ax - b = 0$. Now set $h(x) = f(x) - ax^2/2 - bx$. Since $h'(x) = f'(x) - ax - b = 0$, we have $h(x) = c$ for some constant c. This shows $f(x) = ax^2/2 + bx + c$.

29.8 The set-up for all 3 remaining parts is the same. Consider any x_1, x_2 with $a < x_1 < x_2 < b$. By the mean value theorem, there exists some $x \in (x_1, x_2)$ such that

$$f'(x) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

(ii) $f'(x) < 0$ and $x_1 < x_2$ imply that $f(x_1) > f(x_2)$, so f is strictly decreasing.

(iii) $f'(x) \geq 0$ and $x_1 < x_2$ imply that $f(x_1) \leq f(x_2)$, so f is increasing.

(iv) $f'(x) \leq 0$ and $x_1 < x_2$ imply that $f(x_1) \geq f(x_2)$, so f is decreasing.

29.10

(a)

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\ &= \lim_{x\to 0} \frac{x^2\sin(1/x)+x/2}{x} \\ &= \lim_{x\to 0} x\sin(1/x)+1/2. \end{aligned}$$

Since the limits of both terms exist, and in particular $\lim_{x\to 0} x\sin(1/x) = 0$ and $\lim_{x\to 0} 1/2 = 1/2$, we can use the addition law to get that $f'(0) = 1/2 > 0$.

(b) By corollary 29.7 and assignment 11, a function f is increasing on some interval if and only if $f' \geq 0$ on that interval. Thus, it will suffice to show that in any open interval containing 0, there exists some x with $f'(x) < 0$. First we compute the derivative using the product and chain rules.

$$f'(x) = 2x\sin(1/x) - \cos(1/x) + 1/2.$$

Now consider $y_n = 1/2\pi n$, $n \in \mathbb{N}$. Note that $f'(y_n) = -1/2$ for all n, and that any open interval containing 0 must contain some y_n. This proves that f cannot be increasing on any open interval containing 0.

(c) Although $f'(0) > 0$, the derivative is not positive (or even nonnegative) in any neighborhood of 0 so 29.7 does not apply. This is because the derivative is not continuous at 0.

29.13 Consider the auxiliary function $h(x) = f(x) - g(x)$. Then $h(0) = 0$ and $h'(x) \leq 0$ for all $x \in \mathbb{R}$. By corollary 29.7 h is decreasing on $\mathbb{R}$. In particular, for any $x \geq 0$ we have $h(x) \leq h(0) = 0$. Then $f(x) - g(x) \leq 0$, so $f(x) \leq g(x)$ for all $x \geq 0$.

29.14 This is an application of the previous problem. Let $f_1(x) = x$, so $f_1(0) = 0 = f(0)$ and $f_1'(x) = 1 \leq f'(x)$. By the previous problem, $x \leq f(x)$ for all $x \geq 0$. Now let $f_2(x) = 2x$, so $f_2(0) = 0 = f(0)$ and $f(x) \leq 2 = f_2'(x)$. By the previous problem, $f(x) \leq 2x$ for all $x \geq 0$.