HOMEWORK 11 SOLUTIONS

1.

(a) Suppose that $L = \lim_{x \to x_0} f(x) < 0$. Then consider $\epsilon = L/2$ in the ϵ-δ definition of a limit - the definition states there must exists some $\delta > 0$ such that $|f(x) - L| < \epsilon = L/2$ for all $|x - x_0| < \delta$. Then $f(x) < L/2$ of all $|x - x_0| < \delta$, a contradiction to the non-negativity of f. We conclude that $L \geq 0$.

(b) Suppose that $L = \lim_{x \to x_0} f(x) > 0$. Then consider $\epsilon = L/2$ in the ϵ-δ definition of a limit - the definition states there must exists some $\delta > 0$ such that $|f(x) - L| < \epsilon = L/2$ for all $|x - x_0| < \delta$. Then $f(x) > L/2$ of all $|x - x_0| < \delta$, a contradiction to the non-positivity of f. We conclude that $L \leq 0$.

(c) Choose $x_0 = 0$ and take $f_1(x) = x^2$, $f_2(x) = -x^2$.

2. For any $x < a$, since f is decreasing we have that $f(x) \geq f(a)$. Conversely, for $x > a$ we have $f(x) \leq f(a)$, and therefore

$$g(x) := \frac{f(x) - f(a)}{x - a}$$

satisfies $g(x) \leq 0$ for all $x \neq a$. By question 1, it follows that $\lim_{x \to a} g(x) \leq 0$. Since $\lim_{x \to a} g(x) = f'(a)$, we are done.

The function $f(x) = -x^3$ is strictly decreasing, but has derivative 0 at $x = 0$.

28.2

(a)

$$\begin{aligned}\lim_{x\to 2}\frac{f(x)-f(2)}{x-2} &= \lim_{x\to 2}\frac{x^2-2^3}{x-2}\\ &= \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}\\ &= \lim_{x\to 2}(x^2+2x+4)\\ &= 12.\end{aligned}$$

Evaluation of the final limit follows from continuity.

(b)

$$\begin{aligned}\lim_{x\to a}\frac{g(x)-g(a)}{x-a} &= \lim_{x\to a}\frac{(x+2)-(a+2)}{x-a}\\ &= \lim_{x\to a}\frac{x-a}{x-a}\\ &= \lim_{x\to a} 1\\ &= 1.\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(0)}{x-a} &= \lim_{x\to 0}\frac{x^2\cos x-0}{x-0}\\ &= \lim_{x\to 0} x\cos x\\ &= 0.\end{aligned}$$

Here we apply continuity of $x\cos x$ to evaluate the final limit.

(d)

$$\begin{aligned}\lim_{x\to 1}\frac{r(x)-r(1)}{x-1} &= \lim_{x\to 1}\frac{\frac{3x+4}{2x-1}-7}{x-1}\\ &= \lim_{x\to 1}\frac{11}{1-2x}\\ &= -\frac{11}{2}.\end{aligned}$$

Again we use continuity of $11/(1-2x)$. In particular, this function is continuous at $x=1$ because it is a quotient of continuous functions and the denominator is nonzero at $x=1$.

28.4

(a) Since $1/x$ is differentiable for $x \neq 0$ and sine is differentiable everywhere, by the composition law $\sin(1/x)$ is differentiable for $x \neq 0$. We also have that x^2 is differentiable everywhere, so $x^2 \sin(1/x)$ is differentiable for $x \neq 0$. Using the product and chain rules,

$$f'(a) = 2a \sin\left(\frac{1}{a}\right) - \cos\left(\frac{1}{a}\right).$$

(b)

$$\begin{aligned}\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} \\ &= \lim_{x \to 0} x \sin(1/x) \\ &= 0.\end{aligned}$$

To see this final limit, observe that $\lim_{x \to 0} |x \sin(1/x)| \leq \lim_{x \to 0} |x| = 0$. Then apply the lemma proved in homework 10.2.

(c) By part (b), $f'(0) = 0$. To show that f' is not continuous at 0, it will suffice to demonstrate a sequence x_n converging to zero, but such that $f'(x_n)$ does not converge to 0. Choose $x_n = 1/2n\pi$. Then $\sin(1/x_n) = 0$ and $\cos(1/x_n) = 1$, so we see that

$$\lim_{n \to \infty} f'(x_n) = \lim_{n \to \infty} -1 \neq 0.$$

28.8

(a) The key observation is that $f(x) \leq x^2$ for any x, rational or irrational. Given any $\epsilon > 0$, choose $\delta = \sqrt{\epsilon}$. Then if $|x| < \delta$,

$$|f(x) - f(0)| = |f(x)| \leq |x^2| = |x|^2 < \epsilon.$$

(b) Suppose that $x \neq 0$ is rational. Then for every $n \in \mathbb{N}$, there exists some irrational number $r_n \in (x - 1/n, x + 1/n)$. By construction, this sequence (r_n) converges to x. But $\lim f(r_n) = 0 \neq x^2 = f(x)$.

Now suppose that x is irrational. Then for every $n \in \mathbb{N}$, there exists some rational number $q_n \in (x - 1/n, x + 1/n)$. By construction, this sequence (q_n) converges to x but $\lim f(q_n) = \lim q_n^2 = x^2 \neq 0 = f(x)$.

(c) I claim that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} = 0$. To see this, let $\epsilon > 0$ be given and choose $\delta = \epsilon$. Then for any $|x| < \delta$ (and of course $x \neq 0$),

$$\left|\frac{f(x)}{x} - 0\right| \leq \left|\frac{x^2}{x}\right| = |x| < \delta = \epsilon.$$

We conclude that $f'(0)$ exists and equals 0.

28.14

(a) Consider $g(x) = f(x+a)$, so $f(x) = g(x-a)$. Then

$$\begin{aligned} f'(a) &= g'(0) \\ &= \lim_{h\to 0} \frac{g(h)-g(0)}{h-0} \\ &= \lim_{h\to 0} \frac{f(h+a)-f(a)}{h}. \end{aligned}$$

(b) Since the limit in part (a) exists, both left-hand and right-hand limits also exist and are equal. This implies that we can flip the sign of h, as long as it is done consistently throughout the limit.

$$\begin{aligned} \lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} &= \lim_{h\to 0} \frac{f(a+h)-f(a)+f(a)-f(a-h)}{2h} \\ &= \frac{1}{2} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} + \frac{f(a)-f(a-h)}{h} \\ &= \frac{1}{2} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} + \frac{1}{2} \lim_{h\to 0} \frac{f(a)-f(a-h)}{h} \\ &= \frac{1}{2} f'(a) + \frac{1}{2} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= f'(a). \end{aligned}$$