LIMITS OF FUNCTIONS

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ABSTRACT. Class notes for MAT319, Spring 2012. These complement (but do not subsume or replace!) section 20 of the textbook.

The textbook gives a very general definition for limits of functions, indended to encompass all the possible special cases. For your convenience, we give here the more standard definitions, case by case. Try to see how these fit into textbook's unified scheme!

Intuitively, $\lim_{x\to a} f(x) = L$ means that the values $f(x)$ are close to L when x is close to a. The point a itself is not considered here; $f(a)$ may even be undefined. We only consider $x \neq a$.

Definition 1. *The function f has a (finite or infinite) limit L at the point a (notation: $\lim_{x\to a} f(x) = L$) if for every sequence (x_n) such that $x_n \to a$, $x_n \neq a$, we have that $f(x_n) \to L$. (We always assume that f is defined in a neighborhood of a, excluding perhaps a itself.)*

The limit we just defined is sometimes called "two-sided". One may also be interested in "one-sided" limits that arise when x approaches a from only one side.

Definition 2. *We say that $\lim_{x\to a+} f(x) = L$ if for every sequence (x_n) such that $x_n \to a$, $x_n > a$, we have that $f(x_n) \to L$. (Here, we assume that f is defined on some interval $(a, a+d)$; to left of a, the function may be undefined.)*

The definition of $\lim_{x\to a-} f(x) = L$ is very similar. We consider sequences to the left of a, $x_n < a$. (Make sure you can write a complete definition!)

In the same vein, we can define limits at infinity:

Definition 3. *We say that $\lim_{x\to +\infty} f(x) = L$ if for every sequence (x_n) such that $x_n \to +\infty$, we have that $f(x_n) \to L$. (The function f is required to be defined on some ray $(d, +\infty)$.)*

(Give a definition for limit at $-\infty$.)

Using these definitions, we can test functions for existence of limit, prove limit laws (similar to limit laws for limits of sequences), and some other theorems, such as uniqueness of limits (a function cannot have more than one limit at a given point). We did some of it in class; the rest is a great exercise.

Exercise. State and prove at least some of the above theorems.

Sometimes it useful to articulate the idea of closedness in a different way: $\lim_{x\to a} f(x) = L$ if for any ϵ-neighborhood of L given to us, we can find a δ-neighborhood $(a-\delta, a+\delta)$ of a, such that whenever x is in $(a-\delta, a+\delta)$, and $x \neq a$, we have that $f(x)$ is in the ϵ-neighborhood of L. (This works only if L is a finite number, not for $L = \pm\infty$. Why? What are the changes needed for infinite limits?) We can restate this with inequalities:

Definition 4. $\lim_{x\to a} f(x) = L$ *if for every* $\epsilon > 0$ *we can find* $\delta > 0$ *such that whenever* $|x - a| < \delta$ *and* $x \neq a$*, we have that* $|f(x) - L| < \epsilon$*.*

For limits at infinity (say $+\infty$), a δ-neighborhood of a is replaced by a "neighborhood" of $+\infty$, which has the form $(\alpha, +\infty)$. We then have:

Definition 5. $\lim_{x\to+\infty} f(x) = L$ *if for every* $\epsilon > 0$ *we can find* α *such that whenever* $x > \alpha$ *we have that* $|f(x) - L| < \epsilon$*.*

Exercise. State other similar definitions, say for $\lim_{x\to-\infty} f(x) = L$ and $\lim_{x\to a+} f(x) = -\infty$.

Every time we give two different definitions for the same notion, we have to check that the two definitions are equivalent. That is, if a function satisfies the conditions of the first definition, we must make sure that it would also satisfy the second, and vice versa. For $\lim_{x\to a} f(x)$, we checked in class that the sequences definition follows from the ϵ-definition.

Exercise. Prove (or find in the book) the converse. Mimic the proof that two definitions of continuous functions are equivalent. Prove similar statements for some of the other flavors of the limits (one-sided limits, limits at infinity.)

Finally, we mention two more important theorems (see the book or your lecture/recitation notes). The proofs are a bit easier if you use ϵ-definition, but can also be obtained via sequences.

Theorem 6. *1) If* $\lim_{x\to a} f(x) = L$*, then both one-sided limits at* a *exist and are equal to* L*.*

2) If both one-sided limits exist and are equal, $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = L$*, then the two-sided limit at* a *exists and equals to* L*.*

Theorem 7. *1) If* f *is continuous at* a*, then* $\lim_{x\to a} f(x)$ *exists and equals* $f(a)$*.*

2) If $f(a)$ *is defined, and* $\lim_{x\to a} f(x)$ *exists and equals* $f(a)$*, then* f *is continuous at* a*.*