MAT 319 Introduction to Analysis

Homework 8

due Thursday, March 29

Question 1. Let $f(x)$ be continuous on $[0,1]$, $g(x)$ continuous on $[1,2]$, and $f(1) = g(1) = 5$. Define a function $h(x)$ on $[0,2]$ by

$$h(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq 1 \\ g(x) & \text{if } 1 < x \leq 2. \end{cases}$$

Prove that $h(x)$ is continuous (a) at $x = 1$ (b) at all other points in $[0,2]$.

Note: this question is a special case of the following theorem: if f is continuous on $[a,b]$, g is continuous on adjacent interval $[b,c]$, and $f(b) = g(b)$, then one can "splice" f and g to make a function h on $[a,c]$, and the resulting function h is continuous on its domain. The general theorem is proved in a very similar way (we restrict to special case only to simplify notation).

Question 2. Let $f(x)$ be continuous on $[0,1]$, $g(x)$ continuous on $[1,2]$. Define a function $h(x)$ on $[0,2]$ by

$$h(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq 1 \\ g(x) & \text{if } 1 < x \leq 2. \end{cases}$$

Suppose that $h(x)$ is continuous on $[0,2]$. Prove that $f(1) = g(1)$.

Question 3. Let $f(x)$ be a continuous function. Define $g(x)$ via

$$g(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise .} \end{cases}$$

Prove, using the ϵ-δ definition, that $g(x)$ is continuous. Use the following strategy.

(a) Suppose that $f(a) > 0$. Show that there is a neighborhood of a where $f(x) > 0$ for all x. Use this to show that g is continuous at a.

(b) Do the case $f(a) < 0$ in a similar way.

(c) Suppose that $f(a) = 0$. Show that for any given $\epsilon > 0$, there is a neighborhood of a where $|f(x)| < \epsilon$. Use this to show that g is continuous at a.

Note for (c): if $f(a) = 0$, the function can be pretty complicated in the neighborhood: there may be points with $f(x) = 0$ and $f(x) \neq 0$ arbitrarily close to a.

Question 4. Does there exist a continuous function $f(x)$ such that

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n} \text{ for every } n?$$

Depending on the approach you take, the proof of your answer may be a bit tedious. You are not required to spell out all details, but please explain your answer as best you can.