1. Let \( f(x) \) be continuous on \([0, 1]\), \( g(x) \) continuous on \([1, 2]\), and \( f(1) = g(1) = 5 \). Define a function \( h(x) \) on \([0, 2]\) by

\[
h(x) = \begin{cases} 
  f(x) & \text{if } 0 \leq x \leq 1 \\
  g(x) & \text{if } 1 < x \leq 2.
\end{cases}
\]

Prove that \( h(x) \) is continuous at

(a) \( x = 1 \).

Let \( \epsilon > 0 \). Since \( f \) is continuous at 1, there exists \( \delta_f > 0 \) such that if \( x \in [0, 1] \) and \( |x - 1| < \delta_f \), then \( |f(x) - 5| < \epsilon \). Similarly, there exists \( \delta_g > 0 \) such that if \( x \in [1, 2] \) and \( |x - 1| < \delta_g \), then \( |g(x) - 5| < \epsilon \). Let \( \delta = \min\{\delta_f, \delta_g\} \), and assume \( x \) is such that \( |x - 1| < \delta \). If \( x \leq 1 \), then

\[
|h(x) - 5| = |f(x) - 5| < \epsilon \quad \text{because} \quad |x - 1| < \delta_f.
\]

If \( x > 1 \), then \( |h(x) - 5| = |g(x) - 5| < \epsilon \) because \( |x - 1| < \delta_g \). Therefore, \( |x - 1| < \delta \implies |h(x) - 5| < \epsilon \). So \( h(x) \) is continuous at 5.

(b) all other points \( x_0 \).

The material point is that continuity is a local property. In mathematics, “local” means it is determined by what happens in sufficiently small neighborhoods. In this case, if \( h \) is identical to a known continuous function in some specific neighborhood of \( x_0 \), then \( h \) must be continuous at \( x_0 \), as well. This fact appears in the proofs below: in order to use the continuity of \( f \) and \( g \) to prove the continuity of \( h \), we need to first restrict our attention to a small neighborhood of \( x_0 \) where \( h \) is either identical to \( f \) or identical to \( g \).

i. Since we used an \((\epsilon - \delta)\)-argument in part (a), let’s show how a sequence argument could be used here. First, we’ll treat the case when \( 0 \leq x_0 < 1 \). Let \( (x_n) \) be a sequence in \([0, 2]\) converging to \( x_0 \). Since \( x_0 < 1 \), we can find an \( N \)-tail of \( (x_n) \) that is always less than 1. Therefore, for \( n > N \), \( h(x_n) = f(x_n) \), and by the continuity of \( f \) at \( x_0 \), this tail converges to \( f(x_0) = h(x_0) \). Hence, \( h \) is convergent at \( x_0 \) whenever \( x_0 < 1 \).

The argument is analogous when \( 1 < x_0 \leq 2 \). Let \( (x_n) \) be a sequence in \([0, 2]\) converging to \( x_0 \). There must be an \( M \)-tail that is always greater than 1. Restricting to this tail, \( h(x_n) = g(x_n) \), and since \( g \) is continuous at \( x_0 \), \( g(x_n) \rightarrow g(x_0) = h(x_0) \). Thus, \( h(x_n) \) converges to \( h(x_0) \), and \( h \) is continuous at \( x_0 \).

ii. For those who prefer \( \epsilon \)'s and \( \delta \)'s, suppose first that \( 0 \leq x_0 < 1 \). Let \( \epsilon > 0 \). Since \( f \) is continuous at \( x_0 \), there exists \( \delta_f > 0 \) such that if \( |x - x_0| < \delta_f \) and \( 0 \leq x \leq 1 \) (because this is the domain of \( f \)), then \( |f(x) - f(x_0)| < \epsilon \). Since \( h \) is only identical to \( f \) for \( x \leq 1 \), let 

\[
\delta = \min\{\delta_f, 1 - x_0\}.
\]

The purpose of the extra restriction is the following:

\[
|x - x_0| < \delta \implies |x - x_0| < 1 - x_0 \implies x - x_0 < 1 - x_0 \implies x < 1 \implies h(x) = f(x).
\]

Therefore, if \( |x - x_0| < \delta \) and \( x \in \text{domain}(h) \), then \( |h(x) - h(x_0)| = |f(x) - f(x_0)| < \epsilon \).

Similarly, if \( 1 < x_0 \leq 2 \), then there exists \( \delta_g > 0 \) such that if \( |x - x_0| < \delta_g \) and \( 1 \leq x \leq 2 \), then \( |g(x) - g(x_0)| < \epsilon \). Set \( \delta = \min\{\delta_g, x_0 - 1\} \). Therefore, if \( |x - x_0| < \delta \), then \( 1 < x \), and so

\[
|h(x) - h(x_0)| = |g(x) - g(x_0)| < \epsilon.
\]

2. Let \( f(x) \) be continuous on \([0, 1]\), \( g(x) \) continuous on \([1, 2]\). Define a function \( h(x) \) on \([0, 2]\) by

\[
h(x) = \begin{cases} 
  f(x) & \text{if } 0 \leq x \leq 1 \\
  g(x) & \text{if } 1 < x \leq 2.
\end{cases}
\]

Suppose that \( h(x) \) is continuous on \([0, 2]\). Prove that \( f(1) = g(1) \).

Let \( (x_n) \) be a sequence converging to \( 1 \) such that every term is strictly greater than 1. If you like to
3. Let $f$ be a continuous function. Define $g(x)$ via

$$
g(x) = \begin{cases} 
  f(x) & \text{if } f(x) > 0 \\
  0 & \text{otherwise.}
\end{cases}
$$

Prove, using the $\epsilon - \delta$ definition, that $g(x)$ is continuous. Use the following strategy. Let $a$ be arbitrary. We will divide the proof into cases based upon the value of $f(a)$.

(a) Suppose that $f(a) > 0$. Since $f$ is continuous at $a$, there exists $\delta_1 > 0$ such that if $|x - a| < \delta_1$, then $|f(x) - f(a)| < f(a)$. In particular, this implies that $f(x) > 0$ in this neighborhood of $a$. Hence, for all $|x - a| < \delta_1$, $g(x) = f(x)$. Now let $\epsilon > 0$. Again by the continuity of $f$, we can find $\delta_2 > 0$ such that if $|x - a| < \delta_2$, $|f(x) - f(a)| < \epsilon$. Set $\delta = \min \delta_1, \delta_2$, and assume $|x - a| < \delta$. Consecutively using the facts that $|x - a|$ is less than both $\delta_1$ and $\delta_2$, we can conclude that $|g(x) - g(a)| = |f(x) - f(a)| < \epsilon$.

Therefore, $g$ is continuous at $a$. Like the proofs in question (1b), the proof here relies on the fact that $g$ is identical to $f$ in a neighborhood (given by $\delta_1$) of $a$. Finding this neighborhood reduces the problem to the statement that $f$ is continuous at $a$.

(b) Suppose now that $f(a) < 0$. Then $-f(a) > 0$, and so there exists $\delta > 0$ such that $|f(x) - f(a)| < -f(a)$ whenever $|x - a| < \delta$. Thus, for all $|x - a| < \delta$, $f(x) > 0$, and therefore $g(x) = 0 = g(a)$.

This already implies that $g$ is continuous at $a$; in fact, $g$ is constant in a neighborhood of $a$.

(c) Suppose that $f(a) = 0$. Let $\epsilon > 0$. Since $f$ is continuous at $a$, we can find $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x)| = |f(x) - f(a)| < \epsilon$. Suppose $|x - a| < \delta$. If $f(x) > 0$, then $0 < f(x) < \epsilon$, and so $|g(x)| < \epsilon$. If $f(x) \leq 0$, then $g(x) = 0 < \epsilon$. In either case, $|g(x) - g(a)| = |g(x)| < \epsilon$. Thus, $g$ is continuous at $a$.

4. Does there exist a continuous function $f(x)$ such that

$$
f \left( \frac{1}{n} \right) = \frac{(-1)^n}{n} \text{ for every } n?$$

Yes. If you didn’t understand this problem, try sketching a graph to match the description below.

One way construct such a function is to define $f$ on each interval $\left[ \frac{1}{n+1}, \frac{1}{n} \right]$ so that its graph is the line segment joining the points $\left( \frac{1}{n+1}, \frac{(-1)^{n+1}}{n+1} \right)$ and $\left( \frac{1}{n}, \frac{(-1)^n}{n} \right)$. Since the line segments defined in this way on consecutive intervals $\left[ \frac{1}{n+1}, \frac{1}{n} \right]$ and $\left[ \frac{1}{n}, \frac{1}{n-1} \right]$ have the same value on the overlapping point $x = \frac{1}{n}$, we can “glue” them together like in Problem 1 to get a continuous function on the set

$$
\cup_{n=1}^{\infty} \left[ \frac{1}{n+1}, \frac{1}{n} \right] = (0, 1].
$$

So far, $f$ is defined on $(0, 1]$, and its graph is very jagged. Since $f(1) = -1$ is the point that is farthest to the right, we can extend the graph of $f$ in any continuous way to the right of $x = 1$ as long as it meets up at the point $(-1, 1)$. Probably the easiest way is to simply declare that $f(x) = -1$ for all $x > 1$.

As $n \to \infty$, $f(1/n)$ tends to 0, so we must define $f(0) = 0$ to have a chance of making $f$ continuous everywhere. Therefore, we might as well define $f$ to be identically 0 on $(-\infty, 0)$, thus guaranteeing that $f$ will be continuous at negative values of $x$. 


All that is left is to show that $f$ is continuous at 0. The idea is that since $|f(1/n)| = 1/n$, and that the graph of $f$ zig-zags between its values on the set \( \{ \frac{1}{n} | n \in \mathbb{N} \} \), we can see that in fact, $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$. Proving this fact is a bit technically annoying, but since the purpose of this problem was to help you develop your intuition, I won’t go through the details. The sketch of the graph of this function should convince you that it is correct. Therefore, $f$ is indeed continuous at 0.

If you prefer simple-looking formulas, we can find another function satisfying the desired conditions if we assume some basic properties of cosine:

$$f(x) = \begin{cases} 
  x \cos \left( \frac{\pi}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0.
\end{cases}$$

First, $\frac{\pi}{x}$ is a rational function, and so continuous when $x \neq 0$. $\cos (\pi/x)$ is a composition of continuous functions, and so continuous. $x \cos (\pi/x)$ is a product of continuous functions. This proves that $f$ is continuous at every nonzero point $x$. To show that $f$ is continuous at 0, let $\epsilon > 0$ and take $\delta = \epsilon$. If $|x| < \delta$, then either $x = 0$, and $f(x) = 0 < \epsilon$, or $x \neq 0$ and

$$|f(x)| = \left| x \cos \left( \frac{\pi}{x} \right) \right| \leq |x| < \delta = \epsilon.$$

(We used the fact that $|\cos \theta| \leq 1$ for any $\theta \in \mathbb{R}$.) Thus, $f$ is continuous at 0 as well.