MAT 319, Spring 2012
Solutions to HW 8

1. Let $f(x)$ be continuous on $[0,1]$, $g(x)$ continuous on $[1,2]$, and $f(1) = g(1) = 5$. Define a function $h(x)$ on $[0,2]$ by
$$h(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq 1 \\ g(x) & \text{if } 1 < x \leq 2. \end{cases}$$
Prove that $h(x)$ is continuous at

   (a) $x = 1$.
   Let $\epsilon > 0$. Since f is continuous at 1, there exists $\delta_f > 0$ such that if $x \in [0,1]$ and $|x-1| < \delta_f$, then $|f(x) - 5| < \epsilon$. Similarly, there exists $\delta_g > 0$ such that if $x \in [1,2]$ and $|x-1| < \delta_g$, then $|g(x) - 5| < \epsilon$. Let $\delta = \min \delta_f, \delta_g$, and assume x is such that $|x-1| < \delta$. If $x \leq 1$, then $|h(x) - 5| = |f(x) - 5| < \epsilon$ because $|x-1| < \delta_f$. If $x > 1$, then $|h(x) - 5| = |g(x) - 5| < \epsilon$ because $|x-1| < \delta_g$. Therefore, $|x-1| < \delta \implies |h(x) - 5| < \epsilon$. So $h(x)$ is continuous at 5.

   (b) all other points x_0.
   The material point is that continuity is a local property. In mathematics, "local" means it is determined by what happens in sufficiently small neighborhoods. In this case, if h is identical to a known continuous function in some specific neighborhood of x_0, then h must be continuous at x_0, as well. This fact appears in the proofs below: in order to use the continuity of f and g to prove the continuity of h, we need to first restrict our attention to a small neighborhood of x_0 where h is either identical to f or identical to g.

      i. Since we used an $(\epsilon - \delta)$-argument in part (a), let's show how a sequence argument could be used here. First, we'll treat the case when $0 \leq x_0 < 1$. Let (x_n) be a sequence in $[0,2]$ converging to x_0. Since $x_0 < 1$, we can find an N-tail of (x_n) that is always less than 1. Therefore, for $n > N$, $h(x_n) = f(x_n)$, and by the continuity of f at x_0, this tail converges to $f(x_0) = h(x_0)$. Hence, h is convergent at x_0 whenever $x_0 < 1$.
      The argument is analogous when $1 < x_0 \leq 2$. Let (x_n) be a sequence in $[0,2]$ converging to x_0. There must be an M-tail that is always greater than 1. Restricting to this tail, $h(x_n) = g(x_n)$, and since g is continuous at x_0, $g(x_n) \to g(x_0) = h(x_0)$. Thus, $h(x_n)$ converges to $h(x_0)$, and h is continuous at x_0.

      ii. For those who prefer ϵ's and δ's, suppose first that $0 \leq x_0 < 1$. Let $\epsilon > 0$. Since f is continuous at x_0, there exists $\delta_f > 0$ such that if $|x - x_0| < \delta_f$ and $0 \leq x \leq 1$ (because this is the domain of f), then $|f(x) - f(x_0)| < \epsilon$. Since h is only identical to f for $x \leq 1$, let $\delta = \min\{\delta_f, 1 - x_0\}$. The purpose of the extra restriction is the following:
$$\begin{aligned} |x - x_0| < \delta &\implies |x - x_0| < 1 - x_0 \\ &\implies x - x_0 < 1 - x_0 \\ &\implies x < 1 \\ &\implies h(x) = f(x). \end{aligned}$$
      Therefore, if $|x - x_0| < \delta$ and $x \in \mathit{domain}(h)$, then $|h(x) - h(x_0)| = |f(x) - f(x_0)| < \epsilon$.
      Similarly, if $1 < x_0 \leq 2$, then there exists $\delta_g > 0$ such that if $|x - x_0| < \delta_g$ and $1 \leq x \leq 2$, then $|g(x) - g(x_0)| < \epsilon$. Set $\delta = \min\{\delta_g, x_0 - 1\}$. Therefore, if $|x - x_0| < \delta$, then $1 < x$, and so
$$|h(x) - h(x_0)| = |g(x) - g(x_0)| < \epsilon.$$

2. Let $f(x)$ be continuous on $[0,1]$, $g(x)$ continuous on $[1,2]$. Define a function $h(x)$ on $[0,2]$ by
$$h(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq 1 \\ g(x) & \text{if } 1 < x \leq 2. \end{cases}$$
Suppose that $h(x)$ is continuous on $[0,2]$. Prove that $f(1) = g(1)$.
Let (x_n) be a sequence converging to 1 such that every term is strictly greater than 1. If you like to

be more specific, you could choose $x_n = 1 + \frac{1}{n}$, as an example. The point is that $h(x_n) = g(x_n)$ for all n, and both h and g are continuous at 1. Therefore, $h(x_n) \to h(1) = f(1)$. But $g(x_n) \to g(1)$. The sequence cannot have two limits, and so $f(1) = g(1)$.

3. Let $f(x)$ be a continuous function. Define $g(x)$ via

$$g(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Prove, using the $\epsilon - \delta$ definition, that $g(x)$ is continuous. Use the following strategy. Let a be arbitrary. We will divide the proof into cases based upon the value of $f(a)$.

(a) Suppose that $f(a) > 0$. Since f is continuous at a, there exists $\delta_1 > 0$ such that if $|x - a| < \delta_1$, then $|f(x) - f(a)| < f(a)$. In particular, this implies that $f(x) > 0$ in this neighborhood of a. Hence, for all $|x - a| < \delta_1$, $g(x) = f(x)$. Now let $\epsilon > 0$. Again by the continuity of f, we can find $\delta_2 > 0$ such that if $|x - a| < \delta_2$, $|f(x) - f(a)| < \epsilon$. Set $\delta = \min \delta_1, \delta_2$, and assume $|x - a| < \delta$. Consecutively using the facts that $|x - a|$ is less than both δ_1 and δ_2, we can conclude that

$$|g(x) - g(a)| = |f(x) - f(a)| < \epsilon.$$

Therefore, g is continuous at a.
Like the proofs in question (1b), the proof here relies on the fact that g is identical to a f in a neighborhood (given by δ_1) of a. Finding this neighborhood reduces the problem to the statement that f is continuous at a.

(b) Suppose now that $f(a) < 0$. Then $-f(a) > 0$, and so there exists $\delta > 0$ such that $|f(x) - f(a)| < -f(a)$ whenever $|x - a| < \delta$. Thus, for all $|x - a| < \delta$, $f(x) < 0$, and therefore $g(x) = 0 = g(a)$. This already implies that g is continuous at a; in fact, g is constant in a neighborhood of a.

(c) Suppose that $f(a) = 0$. Let $\epsilon > 0$. Since f is continuous at a, we can find $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x)| = |f(x) - f(a)| < \epsilon$. Suppose $|x - a| < \delta$. If $f(x) > 0$, then $0 < f(x) < \epsilon$, and so $|g(x)| < \epsilon$. If $f(x) \leq 0$, then $g(x) = 0 < \epsilon$. In either case, $|g(x) - g(a)| = |g(x)| < \epsilon$. Thus, g is continuous at a.

4. Does there exist a continuous function $f(x)$ such that

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n} \text{ for every } n?$$

Yes. If you didn't understand this problem, try sketching a graph to match the description below.

One way construct such a function is to define f on each interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ so that its graph is the line segment joining the points $\left(\frac{1}{n+1}, \frac{(-1)^{n+1}}{n+1}\right)$ and $\left(\frac{1}{n}, \frac{(-1)^n}{n}\right)$. Since the line segments defined in this way on consecutive intervals $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ and $\left[\frac{1}{n}, \frac{1}{n-1}\right]$ have the same value on the overlapping point $x = \frac{1}{n}$, we can "glue" them together like in Problem 1 to get a continuous function on the set

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right] = (0, 1].$$

So far, f is defined on $(0, 1]$, and its graph is very jagged. Since $f(1) = -1$ is the point that is farthest to the right, we can extend the graph of f in any continuous way to the right of $x = 1$ as long as it meets up at the point $(-1, 1)$. Probably the easiest way is to simply declare that $f(x) = -1$ for all $x > 1$.
As $n \to \infty$, $f(1/n)$ tends to 0, so we must define $f(0) = 0$ to have a chance of making f continuous everywhere. Therefore, we might as well define f to be identically 0 on $(-\infty, 0]$, thus guaranteeing that f will be continuous at negative values of x.

All that is left is to show that f is continuous at 0. The idea is that since $|f(1/n)| = 1/n$, and that the graph of f zig-zags between its values on the set $\left\{\frac{1}{n} | n \in \mathbb{N}\right\}$, we can see that in fact, $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$. Proving this fact is a bit technically annoying, but since the purpose of this problem was to help you develop your intuition, I won't go through the details. The sketch of the graph of this function should convince you that it is correct. Therefore, f is indeed continuous at 0.

If you prefer simple-looking formulas, we can find another function satisfying the desired conditions if we assume some basic properties of cosine:

$$f(x) = \begin{cases} x\cos\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

First, $\frac{\pi}{x}$ is a rational function, and so continuous when $x \neq 0$. $\cos(\pi/x)$ is a composition of continuous functions, and so continuous. $x\cos(\pi/x)$ is a product of continuous functions. This proves that f is continuous at every nonzero point x. To show that f is continuous at 0, let $\epsilon > 0$ and take $\delta = \epsilon$. If $|x| < \delta$, then either $x = 0$, and $f(x) = 0 < \epsilon$, or $x \neq 0$ and

$$|f(x)| = \left| x\cos\left(\frac{\pi}{x}\right)\right| \leq |x| < \delta = \epsilon.$$

(We used the fact that $|\cos\theta| \leq 1$ for any $\theta \in \mathbb{R}$.) Thus, f is continuous at 0 as well.