1. Prove that the function $f(x) = 4x - 5$ is continuous at every point $x_0$.

(a) using the sequences definition.
Let $(x_n)$ be a sequence converging to $x_0$. $f(x_n) = 4x_n - 5$. Our limit product rule implies that $(4x_n)$ converges to $4x_0$. Our limit sum formula implies that $(4x_n - 5)$ converges to $4x_0 - 5$. Thus, $f(x_n)$ converges to $f(x_0)$, and so $f$ is continuous at $x_0$.

(b) using the $\epsilon - \delta$ definition.
Let $\epsilon > 0$ be arbitrary. We must find a $\delta > 0$ such that if $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

$|f(x) - f(x_0)| = |4x - 5 - (4x_0 - 5)| = 4|x - x_0|.$

Therefore, if we $\delta = \epsilon/4$, that is, if we only consider values of $x$ such that $|x - x_0| < \frac{\epsilon}{4}$, then $|f(x) - f(x_0)| < \epsilon$ as desired.

2. In class, we considered the function $f(x)$, where

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Prove that this function is continuous at $x_0 = -\frac{1}{2}$.

(a) using the sequences definition.
Let $(x_n)$ be a sequence converging to $-\frac{1}{2}$. Then there must be a tail of this sequence that is always negative: there exists $N$ such that if $n > N$, then $x_n < 0$. Therefore, if $n > N$, then $f(x_n) = 0$. The sequence $(f(x_n))$ has a tail that is constantly 0, and so the sequence converges to $0 = f(-\frac{1}{2})$. Therefore, $f$ is continuous at $-\frac{1}{2}$.

(b) using the $\epsilon - \delta$ definition.
Let $\epsilon > 0$, and let $\delta = 1/2$. Suppose that $|x - x_0| < \delta$. Then $|x + \frac{1}{2}| < \frac{\delta}{2}$, so that, in particular,

$$x + \frac{1}{2} < \frac{1}{2}$$

$$x < 0.$$ 

Therefore, since $x < 0$, we know that $f(x) = 0$. Thus, $|f(x) - f(x_0)| = |0 - 0| = 0 < \epsilon$. Therefore, $f$ is continuous at $x_0$.

3. Consider the function $g(x)$, where

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number}. \end{cases}$$

Prove that $g(x)$ is discontinuous at every point.

(a) using the sequences definition.
First we will show that $g$ is discontinuous at every irrational number. Suppose that $x_0$ is irrational. We know that there exists a sequence $(x_n)$ consisting only of rational numbers that converges to $x_0$. But then $g(x_n) = 1$ constantly, and cannot converge to $g(x_0) = 0$.

Now let $x_0$ be a rational number. We know that there is a sequence $(x_n)$ consisting of only irrational numbers converging to $x_0$. But then $g(x_n) = 0$ constantly, and cannot converge to $g(x_0) = 1$.

Therefore, $g(x)$ is discontinuous at every point.
(b) using the $\epsilon - \delta$ definition.

Let $x_0$ be any number. Now must find a value of $\epsilon$ such that no value of $\delta$ "works." Let $\epsilon = \frac{1}{2}$. (In fact, any value smaller than 1 will work.) Let $\delta > 0$ be arbitrary. We must find $x \in (x_0 - \delta, x_0 + \delta)$ such that $|g(x) - g(x_0)| > \frac{1}{2}$. If $x_0$ is irrational, we let $x$ be a rational number in $(x_0 - \delta, x_0 + \delta)$, and if $x_0$ is rational, we choose $x$ to be an irrational number in the same interval. In either case $|g(x) - g(x_0)| = 1 > \frac{1}{2}$, as desired.

4. Consider the function $h(x)$, where

$$h(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Is $h(x)$ continuous at 0? Prove your answer (from any definition).

$h(x)$ is continuous at 0. We will prove this using the $\epsilon - \delta$ definition. Let $\epsilon > 0$, and set $\delta = \sqrt{\epsilon}$.

Suppose $|x| < \delta$. If $x \leq 0$, then $|h(x)| = 0 < \epsilon$. If $x > 0$, then $|h(x)| = |x^2| < \delta^2 = \epsilon$. Therefore, $|x| < \delta \implies |h(x)| < \epsilon$.

Alternatively, we could have used the sequences definition. Let $(x_n)$ be a sequence converging to 0. We must prove that $h(x_n)$ also converges to 0. Let $\epsilon > 0$. We can find a tail of $(x_n)$ such that $|x_n| < \sqrt{\epsilon}$. As $h(x_n)$ is either 0 or $x_n^2$, it will be smaller than $\epsilon$ in this tail. Therefore, $(h(x_n)) \to 0$ as desired.

5. Suppose that $f(x)$ is a continuous function. Prove (from any definition) that the function $7f(x)$ is also continuous.

This is easier to prove with the sequences definition. Let $x_0$ be any number, and let $(x_n)$ be a sequence converging to $x_0$. Since $f$ is continuous, $(f(x_n))$ converges to $f(x_0)$. Therefore, our sequence limit theorem implies that $(7f(x_n))$ converges to $7f(x_0)$. So $7f(x)$ is continuous at $x_0$. Since $x_0$ was arbitrary, $f$ is continuous everywhere.

To use the $\epsilon - \delta$ definition, let $x_0$ be any number and let $\epsilon > 0$. We want to find a $\delta$ such that if $|x - x_0| < \delta$, then $|7f(x) - 7f(x_0)| < \epsilon$. Since $f$ is continuous and $\frac{\epsilon}{7} > 0$, we can find $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{\epsilon}{7}$ whenever $|x - x_0| < \delta$. This immediately implies the desired conclusion.