MAT 319, Spring 2012
Solutions to HW 7

1. Prove that the function $f(x) = 4x - 5$ is continuous at every point x_0.

   (a) using the sequences definition.
   Let (x_n) be a sequence converging to x_0. $f(x_n) = 4x_n - 5$. Our limit product rule implies that $(4x_n)$ converges to $4x_0$. Our limit sum formula implies that $(4x_n - 5)$ converges to $4x_0 - 5$. Thus, $f(x_n)$ converges to $f(x_0)$, and so f is continuous at x_0.

   (b) using the $\epsilon - \delta$ definition.
   Let $\epsilon > 0$ be arbitrary. We must find a $\delta > 0$ such that if $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

$$|f(x) - f(x_0)| = |(4x - 5) - (4x_0 - 5)| = 4|x - x_0|.$$

   Therefore, if we $\delta = \epsilon/4$, that is, if we only consider values of x such that $|x - x_0| < \frac{\epsilon}{4}$, then $|f(x) - f(x_0)| < \epsilon$ as desired.

2. In class, we considered the function $f(x)$, where

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

   Prove that this function is continuous at $x_0 = -\frac{1}{2}$.

   (a) using the sequences definition.
   Let (x_n) be a sequence converging to $-\frac{1}{2}$. Then there must be a tail of this sequence that is always negative: there exists N such that if $n > N$, then $x_n < 0$. Therefore, if $n > N$, then $f(x_n) = 0$. The sequence $(f(x_n))$ has a tail that is constantly 0, and so the sequence converges to $0 = f(-\frac{1}{2})$. Therefore, f is continuous at $-\frac{1}{2}$.

   (b) using the $\epsilon - \delta$ definition.
   Let $\epsilon > 0$, and let $\delta = 1/2$. Suppose that $|x - x_0| < \delta$. Then$\left|x + \frac{1}{2}\right| < \frac{1}{2}$, so that, in particular,

$$\begin{aligned} x + \frac{1}{2} &< \frac{1}{2} \\ x &< 0. \end{aligned}$$

   Therefore, since $x < 0$, we know that $f(x) = 0$. Thus, $|f(x) - f(x_0)| = |0 - 0| = 0 < \epsilon$. Therefore, f is continuous at x_0.

3. Consider the function $g(x)$, where

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number.} \end{cases}$$

   Prove that $g(x)$ is discontinuous at every point,

   (a) using the sequences definition.
   First we will show that g is discontinuous at every irrational number. Suppose that x_0 is irrational. We know that there exists a sequence (x_n) consisting only of rational numbers that converges to x_0. But then $g(x_n) = 1$ constantly, and cannot converge to $g(x_0) = 0$.
   Now let x_0 be a rational number. We know that there is a sequence (x_n) consisting of only irrational numbers converging to x_0. But then $g(x_n) = 0$ constantly, and cannot converge to $g(x_0) = 1$.
   Therefore, $g(x)$ is discontinuous at every point.

(b) using the $\epsilon - \delta$ definition.
Let x_0 be any number. Now must find a value of ϵ such that no value of δ "works." Let $\epsilon = \frac{1}{2}$. (In fact, any value smaller than 1 will work.) Let $\delta > 0$ be arbitrary. We must find $x \in (x_0 - \delta, x_0 + \delta)$ such that $|g(x) - g(x_0)| > \frac{1}{2}$. If x_0 is irrational, we let x be a rational number in $(x_0 - \delta, x_0 + \delta)$, and if x_0 is rational, we choose x to be an irrational number in the same interval. In either case $|g(x) - g(x_0)| = 1 > \frac{1}{2}$, as desired.

4. Consider the function $h(x)$, where

$$h(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Is $h(x)$ continuous at 0? Prove your answer (from any definition).
h is continuous at 0. We will prove this using the $\epsilon - \delta$ definition. Let $\epsilon > 0$, and set $\delta = \sqrt{\epsilon}$. Suppose $|x| < \delta$. If $x \leq 0$, then $|h(x)| = 0 < \epsilon$. If $x > 0$, then $|h(x)| = \left|x^2\right| < \delta^2 = \epsilon$. Therefore, $|x| < \delta \implies |h(x)| < \epsilon$.
Alternatively, we could have used the sequences definition. Let (x_n) be a sequence converging to 0. We must prove that $h(x_n)$ also converges to 0. Let $\epsilon > 0$. We can find a tail of (x_n) such that $|x_n| < \sqrt{\epsilon}$. As $h(x_n)$ is either 0 or x_n^2, it will be smaller than ϵ in this tail. Therefore, $(h(x_n)) \to 0$ as desired.

5. Suppose that $f(x)$ is a continuous function. Prove (from any definition) that the function $7f(x)$ is also continuous.
This is easier to prove with the sequences definition. Let x_0 be any number, and let (x_n) be a sequence converging to x_0. Since f is continuous, $(f(x_n))$ converges to $f(x_0)$. Therefore, our sequence limit theorem implies that $(7f(x_n))$ converges to $7f(x_0)$. So $7f(x)$ is continuous at x_0. Since x_0 was arbitrary, f is continuous everywhere.
To use the $\epsilon - \delta$ definition, let x_0 be any number and let $\epsilon > 0$. We want to find a δ such that if $|x - x_0| < \delta$, then $|7f(x) - 7f(x_0)| < \epsilon$. Since f is continuous and $\frac{\epsilon}{7} > 0$, we can find $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{\epsilon}{7}$ whenever $|x - x_0| < \delta$. This immediately implies the desired conclusion.