MAT 319 Introduction to Analysis

Homework 6

due Wednesday, March 7

Please prove (or explain as appropriate) all your answers.

Question 1. Let (x_n) be a sequence with non-negative terms, $x_n \geq 0$ for all n. Suppose that (x_n) **does not** converge to 0. (It can have a different limit or no limit at all). Prove that one can always find a subsequence (x_{n_k}) such that $x_{n_k} > \epsilon$ for all terms in the subsequence.

Question 2. Suppose that a sequence (x_n) is not bounded below. Prove that it has a subsequence diverging to $-\infty$. Give an example showing that the entire sequence may not diverge to $-\infty$.

(We did a similar question in class for unbounded above. Please give a direct proof here for extra practice.)

Question 3. Suppose that $\sum a_n$ is a series with non-negative terms, $a_n \geq 0$ for all n. Suppose that $\sum a_n$ does not converge. Show that $\sum a_n$ diverges to $+\infty$, that is, for any α there is N such that $s_n > \alpha$ for all partial sums s_n with $n > N$.

Some of the results from previous homeworks (or Question 1 from this homework) may be helpful. A few slightly different solutions are possible.

Question 4. In this question, you'll prove the divergence part of comparison test. Suppose that $\sum a_n$, $\sum b_n$ are two series with non-negative terms, $\sum b_n$ diverges, and $a_n \geq b_n$. Prove that $\sum a_n$ diverges.

Question 5. Please do question 14.14 from the textbook to prove that the harmonic series diverges. Follow the strategy indicated in the book and use the comparison test from Question 3 above. Show carefully that the series $\sum a_n$ described in the book diverges to $+\infty$: prove directly that its partial sums become large.

Question 6. In this question, you'll prove the divergence part of the ratio test. Suppose that $\lim \frac{a_{n+1}}{a_n} = L$, and $L > 1$. Prove that the series $\sum a_n$ diverges. (You can assume that $a_n > 0$ for all n.)