MAT 319, Spring 2012
Solutions to HW 3

1. Find the following limits using limit laws. Explain carefully which theorems you used in each step.

   (a) $x_n = \frac{2n-1}{3n+2}$.
   Multiplying and dividing x_n by $\frac{1}{n}$ is a legitimate algebraic manipulation, so
   $$x_n = \frac{2-\frac{1}{n}}{3+\frac{2}{n}}.$$
   Since $\left(\frac{1}{n}\right)$ converges to 0, Theorem 9.2 tells us that $\left(-\frac{1}{n}\right)$ and $\left(\frac{2}{n}\right)$ converge to 0 as well, since they are multiples of $\left(\frac{1}{n}\right)$. Theorem 9.3 tells us that if we add the constant sequence (2) to $\left(-\frac{1}{n}\right)$, we get the sequence $\left(2-\frac{1}{n}\right)$, which must converge to $2+0=2$. Similarly, $\left(3+\frac{2}{n}\right)$ converges to 3. Finally, since $\left(3+\frac{2}{n}\right)$ is never zero, and converges to a nonzero limit, we can apply Theorem 9.6 to arrive at the conclusion:
   $$\lim\left(x_n\right) = \frac{\lim\left(2-\frac{1}{n}\right)}{\lim\left(3+\frac{2}{n}\right)} = \frac{2}{3}.$$

   (b) $x_n = \frac{7n^3-n^2+1}{2n+5n^3-3}$.
   Now, we miltiply and divide x_n by $\frac{1}{n^3}$:
   $$x_n = \frac{7-\frac{1}{n}+\frac{1}{n^3}}{\frac{2}{n^2}+5-\frac{3}{n^3}}.$$
   Theorem 9.7(a) tells us that $\left(\frac{1}{n}\right)$,$\left(\frac{1}{n^2}\right)$, and $\left(\frac{1}{n^3}\right)$ converge to 0. Theorem 9.2 now tells us that $\left(-\frac{1}{n}\right)$, $\left(\frac{2}{n^2}\right)$, and $\left(-\frac{3}{n^3}\right)$ all converge to 0. Two applications of theorem 9.3 shows that $\left(7-\frac{1}{n}+\frac{1}{n^3}\right)$ converges to 7 and $\left(\frac{2}{n^2}+5-\frac{3}{n^3}\right)$ converges to 5. In order to apply Theorem 9.6, we need to show that $\frac{2}{n^2}+5-\frac{3}{n^3} = \frac{1}{n^3}\left(2n+5n^3-3\right)$ is not 0 for any n; we already know it converges to a nonzero limit. We see that $5n^3+2n-3 > 2n-3 > 0$ if $n \geq 2$, and the $n=1$ case can be checked by hand. Therefore, 9.6 applies, and
   $$\lim\left(x_n\right) = \frac{7}{5}.$$

   (c) $x_n = \frac{n}{n^4+n^3+n^2-n+1}$.
   We multiply and divide x_n by $\frac{1}{n^4}$
   $$x_n = \frac{\frac{1}{n^3}}{1+\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}+\frac{1}{n^4}}.$$
   According to Example 9.7(a), $\left(\frac{1}{n}\right), \left(\frac{1}{n^2}\right), \left(\frac{1}{n^3}\right), \left(\frac{1}{n^4}\right)$ all converge to 0. Therefore, by 9.2 and repeated applications of 9.3,
   $$\lim\left(1+\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}+\frac{1}{n^4}\right) = 1+0+0+0+0 = 1.$$
   Also, $1+\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}+\frac{1}{n^4}$ is positive for all n. This is easy to see since $1-\frac{1}{n^3} \geq 0$ and all other terms are positive. Therefore, we can apply Theorem 9.6 to get
   $$\lim\left(x_n\right) = \frac{\lim\left(\frac{1}{n^3}\right)}{\lim\left(1+\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}+\frac{1}{n^4}\right)} = \frac{0}{1} = 0.$$

2.

(a) Suppose (x_n) converges and (y_n) does not. Prove that $(x_n + y_n)$ diverges.
The easiest proof is probably the following: Assume for sake of contradiction that $(x_n + y_n)$ converges. Since (x_n) converges, so does $(-x_n)$. Therefore, the limit sum theorem tells us that $(x_n + y_n) + (-x_n) = (x_n + y_n - x_n)$ converges. But this is just the sequence (y_n), which we know diverges. Contradiction.
A direct proof is much more difficult; one approach uses the consequence of the triangle inequality that we discussed in recitation: namely

$$|a - b| \geq ||a| - |b|| .$$

In order to show that $(x_n + y_n)$ diverges, we must demonstrate that no number L is a limit. To that end, let A denote $\lim(x_n)$. Since (y_n) is divergent, there is an ϵ-neighborhood of $L - A$ that contains no tail of (y_n). Let us take this ϵ, and use it to find a tail of (x_n) that is contained in the $\frac{\epsilon}{2}$-neighborhood of A. That is, for some number N, $|A - x_n| < \frac{\epsilon}{2}$ whenever $n > N$. But we can find infinitely many $n > N$ such that $|y_n - (L - A)| > \epsilon$. Notice that for these values of n,

$$|y_n - (L - A)| - |A - x_n| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2} > 0.$$

Thus,

$$|(x_n + y_n) - L| = |(y_n - (L - A)) - (A - x_n)| \geq ||y_n - (L - A)| - |A - x_n|| > \frac{\epsilon}{2}.$$

Therefore, there is no tail of $(x_n + y_n)$ contained on the $\epsilon/2$-neighborhood of L. So L is not the limit. As L is arbitrary, $(x_n + y_n)$ has no limit; it diverges.

(b) Suppose both the sequences (x_n) and (y_n) diverge. It is possible that $(x_n + y_n)$ converges. Here are three such examples:

i. Suppose $x_n = n$ and $y_n = -n$. Then $(x_n + y_n)$ is the constant 0 sequence. So it converges to 0.

ii. This example demonstrates that x_n and y_n do not have to tend toward $\pm\infty$ as in the last example.
Let $(x_n) = 1, 0, 1, 0, 1, 0, 1, 0, \dots$ and let $(y_n) = 0, 1, 0, 1, 0, 1, 0, 1, \dots$. Both (x_n) and (y_n) diverge (see problem #5 of last week's homework), but $(x_n + y_n)$ is the constant sequence 1.

iii. Finally, here is an example that might seem less artificial: Let $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$ and let $y_n = -\frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n+1} = -\sum_{k=1}^{n} \frac{1}{k+1}$. We have not shown that (x_n) and (y_n) diverge yet, but it is true. However

$$x_n + y_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{n+1},$$

so that $(x_n + y_n)$ converges to 1. This is the prototypical example of a "telescoping series," which you learned about in calculus. Most of the telescoping series you encountered were the difference of two diverging series.

3. Suppose that (x_n) converges to 4. Show that (x_n) is ultimately nonnegative, so that for sufficiently large n, $\sqrt{x_n}$ is defined. Prove that $\left(\sqrt{x_n}\right)$ converges to 2.
Idea: We know that (x_n) converges to 4, so by looking at sufficiently large tails, we can make the distance between x_n and 4 as small as we like. In order to make sure that x_n is positive, x_n must be closer to 4 than 0 is to 4. Thus, we want a tail that ensures $|x_n - 4| < |0 - 4| = 4$.
Proof: Let N be large enough so that if $n > N$, then $|x_n - 4| < 4$. Therefore, assuming $n > N$, we can conclude that $-4 < x_n - 4 < 4$ and hence $0 < x_n < 8$. In particular, x_n must be positive. Therefore, if $n > N$, $\sqrt{x_n}$ is defined.
To show that $\sqrt{x_n}$ converges to 2, we use the identity:

$$|\sqrt{x_n} - 2| = \left| \frac{x_n - 4}{\sqrt{x_n} + 2} \right| .$$

Since the denominator is always positive, and in fact, always greater than 1, we can conclude

$$|\sqrt{x_n} - 2| < |x_n - 4|\,.$$

Let $\epsilon > 0$. We can find a tail starting at M such that if $n > M$, then $|x_n - 4| < \epsilon$. Therefore, if n is larger than both N and M, (so $n > \max(M, N)$), then

$$|\sqrt{x_n} - 2| < |x_n - 4| < \epsilon.$$

4.

(a) Suppose that the sequence (x_n) converges to A, the sequence (y_n) converges to B, and $x_n \leq y_n$ for all n. Prove that $A \leq B$.
The easiest way to prove this is by contradiction. There is also a more direct proof, but it is much more difficult to grasp.

i. Contradiction Proof: Suppose $B < A$. We must find a specific n such that $y_n < x_n$. In order to do this, set $\epsilon = \frac{A-B}{2} > 0$. We pick ϵ this way so that the ϵ-neighborhoods around A and B are disjoint. Smaller values of ϵ would work just as well. There is a tail of (x_n) that is in the ϵ-neighborhood around A, and a tail of (y_n) that is in the ϵ-neighborhood around B. Pick n large enough so that x_n and y_n are both in their corresponding tail. Then $A-\epsilon < x_n < A+\epsilon$, and $B - \epsilon < y_n < B + \epsilon$. Therefore,

$$y_n < B + \epsilon = \frac{A+B}{2} = A - \epsilon < x_n,$$

and we have our contradiction. If you picked ϵ smaller than $\frac{A-B}{2}$, then you would have found $y_n < B + \epsilon < A - \epsilon < x_n$.

ii. Direct proof: In this proof, we will show that $A < B + \epsilon$ for any positive ϵ, and thus, A must be less than or equal to B. To this end, let $\epsilon > 0$ be arbitrary. We can find a tail of (x_n) inside the $\frac{\epsilon}{2}$-neighborhood of A and a tail of (y_n) inside the $\frac{\epsilon}{2}$-neighborhood of B. Choose n large enough so that x_n and y_n both belong to their corresponding tail. Then $A - \frac{\epsilon}{2} < x_n \leq y_n < B + \frac{\epsilon}{2}$. So that $A < B + \epsilon$.

(b) Suppose that, as in part (a), $x_n \to A$ and $y_n \to B$, but now $x_n < y_n$ for every n. Give an example showing that $A = B$ is still possible.
One easy example is the following: Let $x_n = -\frac{1}{n}$ and let $y_n = \frac{1}{n}$. Then $x_n < 0 < y_n$ for every n, and both sequences converge to 0.