1. Find the following limits using limit laws. Explain carefully which theorems you used in each step.

(a) \( x_n = \frac{2n-1}{3n^2+2} \).

Multiplying and dividing \( x_n \) by \( \frac{1}{n} \) is a legitimate algebraic manipulation, so

\[
x_n = \frac{2 - \frac{1}{n}}{3 + \frac{2}{n}}.
\]

Since \( \left( \frac{1}{n} \right) \) converges to 0, Theorem 9.2 tells us that \( \left( \frac{-1}{n} \right) \) and \( \left( \frac{2}{n} \right) \) converge to 0 as well, since they are multiples of \( \left( \frac{1}{n} \right) \). Theorem 9.3 tells us that if we add the constant sequence \( (2) \) to \( \left( \frac{-1}{n} \right) \), we get the sequence \( \left( 2 - \frac{1}{n} \right) \), which must converge to \( 2 + 0 = 2 \). Similarly, \( (3 + \frac{2}{n}) \) converges to 3. Finally, since \( (3 + \frac{2}{n}) \) is never zero, and converges to a nonzero limit, we can apply Theorem 9.6 to arrive at the conclusion:

\[
\lim (x_n) = \lim \left( \frac{2 - \frac{1}{n}}{3 + \frac{2}{n}} \right) = \frac{2}{3}.
\]

(b) \( x_n = \frac{7n^3 - n^2 + 1}{2n^3 + 5n^2 - 3} \).

Now, we multiply and divide \( x_n \) by \( \frac{1}{n^3} \):

\[
x_n = \frac{7 - \frac{1}{n} + \frac{1}{n^2}}{\frac{2}{n^2} + 5 - \frac{3}{n^3}}.
\]

According to Example 9.7(a), \( \left( \frac{1}{n} \right) \), \( \left( \frac{1}{n^2} \right) \), and \( \left( \frac{1}{n^3} \right) \) all converge to 0. Therefore, by Theorem 9.6, we need to show that \( \frac{2}{n^2} + 5 - \frac{3}{n^3} = \frac{1}{n^3} (2n + 5n^3 - 3) \) is not 0 for any \( n \); we already know it converges to a nonzero limit. We see that \( 5n^3 + 3 > 2n - 3 > 0 \) if \( n \geq 2 \), and the \( n = 1 \) case can be checked by hand. Therefore, 9.6 applies, and

\[
\lim (x_n) = \frac{7}{5}.
\]

(c) \( x_n = \frac{n}{n^4 + n^3 + n^2 - n + 1} \).

We multiply and divide \( x_n \) by \( \frac{1}{n^4} \):

\[
x_n = \frac{\frac{1}{n^4}}{1 + \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4}}.
\]

According to Example 9.7(a), \( \left( \frac{1}{n} \right) \), \( \left( \frac{1}{n^2} \right) \), \( \left( \frac{1}{n^3} \right) \), \( \left( \frac{1}{n^4} \right) \) all converge to 0. Therefore, by Theorem 9.2 and repeated applications of 9.3,

\[
\lim \left( 1 + \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4} \right) = 1 + 0 + 0 + 0 + 0 = 1.
\]

Also, \( 1 + \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4} \) is positive for all \( n \). This is easy to see since \( 1 - \frac{1}{n^4} \geq 0 \) and all other terms are positive. Therefore, we can apply Theorem 9.6 to get

\[
\lim (x_n) = \lim \left( \frac{\frac{1}{n^4}}{1 + \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4}} \right) = \frac{0}{1} = 0.
\]

2.
(a) Suppose \((x_n)\) converges and \((y_n)\) does not. Prove that \((x_n + y_n)\) diverges. The easiest proof is probably the following: Assume for sake of contradiction that \((x_n + y_n)\) converges. Since \((x_n)\) converges, so does \((-x_n)\). Therefore, the limit sum theorem tells us that \((x_n + y_n) + (-x_n) = (x_n + y_n - x_n)\) converges. But this is just the sequence \((y_n)\), which we know diverges. Contradiction.

A direct proof is much more difficult; one approach uses the consequence of the triangle inequality that we discussed in recitation: namely

\[
|a - b| \geq ||a| - |b||.
\]

In order to show that \((x_n + y_n)\) diverges, we must demonstrate that no number \(L\) is a limit. To that end, let \(A\) denote \(\lim (x_n)\). Since \((y_n)\) is divergent, there is an \(\epsilon\)-neighborhood of \(L - A\) that contains no tail of \((y_n)\). Let us take this \(\epsilon\), and use it to find a tail of \((x_n)\) that is contained in the \(\frac{\epsilon}{2}\)-neighborhood of \(A\). That is, for some number \(N\), \(|A - x_n| < \frac{\epsilon}{2}\) whenever \(n > N\). But we can find infinitely many \(n > N\) such that \(|y_n - (L - A)| > \epsilon\). Notice that for these values of \(n\),

\[
|y_n - (L - A)| - |A - x_n| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2} > 0.
\]

Thus,

\[
|(x_n + y_n) - L| = |(y_n - (L - A)) - (A - x_n)| \geq |y_n - (L - A)| - |A - x_n| > \frac{\epsilon}{2}.
\]

Therefore, there is no tail of \((x_n + y_n)\) contained on the \(\epsilon/2\)-neighborhood of \(L\). So \(L\) is not the limit. As \(L\) is arbitrary, \((x_n + y_n)\) has no limit; it diverges.

(b) Suppose both the sequences \((x_n)\) and \((y_n)\) diverge. It is possible that \((x_n + y_n)\) converges. Here are three such examples:

i. Suppose \(x_n = n\) and \(y_n = -n\). Then \((x_n + y_n)\) is the constant 0 sequence. So it converges to 0.

ii. This example demonstrates that \(x_n\) and \(y_n\) do not have to tend toward \(\pm\infty\) as in the last example. Let \((x_n) = 1, 0, 1, 0, 1, 0, 1, 0, \ldots\) and let \((y_n) = 0, 1, 0, 1, 0, 1, 0, 1, \ldots\). Both \((x_n)\) and \((y_n)\) diverge (see problem #5 of last week’s homework), but \((x_n + y_n)\) is the constant sequence 1.

iii. Finally, here is an example that might seem less artificial: Let \(x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}\) and let \(y_n = -\frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n+1} = -\sum_{k=1}^{n} \frac{1}{k+1}\). We have not shown that \((x_n)\) and \((y_n)\) diverge yet, but it is true. However

\[
x_n + y_n = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{n+1},
\]

so that \((x_n + y_n)\) converges to 1. This is the prototypical example of a “telescoping series,” which you learned about in calculus. Most of the telescoping series you encountered were the difference of two diverging series.

3. Suppose that \((x_n)\) converges to 4. Show that \((x_n)\) is ultimately nonnegative, so that for sufficiently large \(n\), \(\sqrt{x_n}\) is defined. Prove that \((\sqrt{x_n})\) converges to 2.

Idea: We know that \((x_n)\) converges to 4, so by looking at sufficiently large tails, we can make the distance between \(x_n\) and 4 as small as we like. In order to make sure that \(x_n\) is positive, \(x_n\) must be closer to 4 than 0 is to 4. Thus, we want a tail that ensures \(|x_n - 4| < |0 - 4| = 4\).

Proof: Let \(N\) be large enough so that if \(n > N\), then \(|x_n - 4| < 4\). Therefore, assuming \(n > N\), we can conclude that \(-4 < x_n - 4 < 4\) and hence \(0 < x_n < 8\). In particular, \(x_n\) must be positive. Therefore, if \(n > N\), \(\sqrt{x_n}\) is defined.

To show that \(\sqrt{x_n}\) converges to 2, we use the identity:

\[
|\sqrt{x_n} - 2| = \left| \frac{x_n - 4}{\sqrt{x_n} + 2} \right|.
\]
Since the denominator is always positive, and in fact, always greater than 1, we can conclude

\[ |\sqrt{x_n} - 2| < |x_n - 4|. \]

Let \( \epsilon > 0 \). We can find a tail starting at \( M \) such that if \( n > M \), then \( |x_n - 4| < \epsilon \). Therefore, if \( n \) is larger than both \( N \) and \( M \), (so \( n > \max(M, N) \)), then

\[ |\sqrt{x_n} - 2| < |x_n - 4| < \epsilon. \]

4.

(a) Suppose that the sequence \( (x_n) \) converges to \( A \), the sequence \( (y_n) \) converges to \( B \), and \( x_n \leq y_n \) for all \( n \). Prove that \( A \leq B \).

The easiest way to prove this is by contradiction. There is also a more direct proof, but it is much more difficult to grasp.

i. Contradiction Proof: Suppose \( B < A \). We must find a specific \( n \) such that \( y_n < x_n \). In order to do this, set \( \epsilon = \frac{A - B}{2} > 0 \). We pick \( \epsilon \) this way so that the \( \epsilon \)-neighborhoods around \( A \) and \( B \) are disjoint. Smaller values of \( \epsilon \) would work just as well. There is a tail of \( (x_n) \) that is in the \( \epsilon \)-neighborhood around \( A \), and a tail of \( (y_n) \) that is in the \( \epsilon \)-neighborhood around \( B \). Pick \( n \) large enough so that \( x_n \) and \( y_n \) are both in their corresponding tail. Then \( A - \epsilon < x_n < A + \epsilon \), and \( B - \epsilon < y_n < B + \epsilon \). Therefore,

\[ y_n < B + \epsilon = \frac{A + B}{2} = A - \epsilon < x_n, \]

and we have our contradiction. If you picked \( \epsilon \) smaller than \( \frac{A - B}{2} \), then you would have found \( y_n < B + \epsilon < A - \epsilon < x_n \).

ii. Direct proof: In this proof, we will show that \( A < B + \epsilon \) for any positive \( \epsilon \), and thus, \( A \) must be less than or equal to \( B \). To this end, let \( \epsilon > 0 \) be arbitrary. We can find a tail of \( (x_n) \) inside the \( \frac{\epsilon}{2} \)-neighborhood of \( A \) and a tail of \( (y_n) \) inside the \( \frac{\epsilon}{2} \)-neighborhood of \( B \). Choose \( n \) large enough so that \( x_n \) and \( y_n \) both belong to their corresponding tail. Then \( A - \frac{\epsilon}{2} < x_n \leq y_n < B + \frac{\epsilon}{2} \). So that \( A < B + \epsilon \).

(b) Suppose that, as in part (a), \( x_n \to A \) and \( y_n \to B \), but now \( x_n < y_n \) for every \( n \). Give an example showing that \( A = B \) is still possible.

One easy example is the following: Let \( x_n = -\frac{1}{n} \) and let \( y_n = \frac{1}{n} \). Then \( x_n < 0 < y_n \) for every \( n \), and both sequences converge to 0.