MAT 319, Spring 2012
Solutions to HW 2

1. Find the limit of the sequence

$$x_n = \frac{2n-1}{3n+2}$$

and prove your answer.

The limit is 2/3. To prove this, let $\epsilon > 0$. Then for all n,

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| = \left|\frac{6n-3-(6n+4)}{9n+6}\right| = \left|\frac{-7}{9n+6}\right| = \frac{7}{9n+6} \leq \frac{7}{9n} < \frac{1}{n}.$$

Therefore, if we restrict to the tail where $n > \frac{1}{\epsilon}$, so that $\frac{1}{n} < \epsilon$, then

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \epsilon.$$

Of course, this is not the only estimate you can make. If you choose, you can solve for n exactly.

2. Find the limit of the sequence

$$x_n = \frac{n^2}{n^2+1}$$

and prove your answer.

The limit is 1. To prove this, let $\epsilon > 0$. Then for all n,

$$\left|\frac{n^2}{n^2+1} - 1\right| = \left|\frac{-1}{n^2+1}\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n}.$$

Therefore, if we restrict to the tail where $n > \frac{1}{\epsilon}$, so that $\frac{1}{n} < \epsilon$, then

$$\left|\frac{n^2}{n^2+1} - 1\right| < \epsilon.$$

Just as in problem #1, this is not the only estimate you can make. For example, you could stop at the $\dfrac{1}{n^2}$ step and take $n > \sqrt{1/\epsilon}$. Alternately, you could solve for n exactly, as well (assuming that $\epsilon \leq 1$).

3. Find the limit of the sequence

$$x_n = \frac{n}{n^4+n^3+n^2-n+1}$$

and prove your answer.

The limit is 0. Let $\epsilon > 0$. First notice that $n^2 \geq n$, so that $n^2 - n$ is nonnegative. This implies $n^4 + n^3 + n^2 - n + 1 > n^3 + n^2 - n + 1 > 0$. Hence,

$$\left|\frac{n}{n^4+n^3+n^2-n+1}\right| = \frac{n}{n^4+n^3+n^2-n+1} < \frac{n}{n^4} = \frac{1}{n^3} \leq \frac{1}{n}.$$

Therefore, if we restrict to the tail where $n > \frac{1}{\epsilon}$, so that $\frac{1}{n} < \epsilon$, then

$$\left|\frac{n}{n^4+n^3+n^2-n+1}\right| < \epsilon.$$

This is only one of many estimates that works in this problem. For example, you could instead estimate it in such a way to leave only n^3 or n^2 in the denominator. The final estimation to reduce it to $1/n$ is not necessary, either.

4. Suppose that the sequence (x_n) converges to A. Working from the definition of the limit, prove that the sequence (y_n), where $y_n = 2x_n + 5$ converges to $2A + 5$.

   Let $\epsilon > 0$. We can find N such that if $n > N$, then $|x_n - A| < \frac{\epsilon}{2}$. Then

$$\begin{aligned} |y_n - (2A+5)| &= |2x_n + 5 - 2A - 5| \\ &= 2\,|x_n - A| \\ &< 2\frac{\epsilon}{2} = \epsilon. \end{aligned}$$

5. Let (x_n) converge to A and (y_n) converge to B, where $A \neq B$. Consider a new sequence where x_n's and y_n's alternate:

$$(a_k) := x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, \ldots$$

   Prove that this sequence diverges.

   There are at least two methods to prove this claim. The first is to prove the contrapositive statement: Assume that (a_k) converges to a limit L, and then prove that (x_n) and (y_n) also converge to L; i.e. $A = B$. In some books, this is method is called an indirect proof. Alternatively, one could assume that $A \neq B$, that (a_k) converges, and argue to a contradiction. We will explore both of these angles below.

   (a) (Indirect proof.) Suppose (a_k) converges to some limit L. Let $\epsilon > 0$. Some N-tail of (a_k) is contained in the ϵ-neighborhood of L. Since $x_n = a_{2n-1}$ and $y_n = a_{2n}$, and $2n > 2n - 1 \geq n$, we see that the N-tails of both (x_n) and (y_n) are contained in the N-tail of (a_k), whence in the ϵ-neighborhood of L. To recap, we have shown that for any $\epsilon > 0$, there exists an N-tail of (x_n), and of (y_n), contained in the ϵ-neighborhood of L. In other words, L is the limit of both (x_n) and (y_n). Since the limit of a sequence is unique, we know $A = L = B$.

   (b) (Proof by contradiction.) Assume that (x_n) converges to A, (y_n) converges to B, and $A \neq B$. Suppose for the sake of contradiction that (a_k) converged to some limit L. There are a couple different ways to proceed from here, but they all will rely on the triangle inequality at some point. The idea is that the distance between A and B is not zero, so we can pick $\epsilon > 0$ so small so as to make A and B appear very far apart. The x's will have to gather around A, the y's around B, and so they will not be able to be close to one another. With this strategy in mind, let $\epsilon = |B - A|/10$. (The factor of 10 is somewhat arbitrary. We just need something large.) Let's take a look at the different methods for reaching a contradiction:

      i. (Case by case, based on the Value for L.)
         Case $L = A$: Since B is the limit of (y_n), an N-tail of (y_n) completely contained in the ϵ-neighborhood of B. Since ϵ is so small, the ϵ-neighborhoods of B and A are disjoint. Therefore, this N-tail of (y_n) exists completely outside of the ϵ-neighborhood of L. Hence, a_{2n} is not contained in the ϵ-neighborhood of L for any $n > N$. But this means that this particular ϵ-neighborhood of L contains no tail of (a_k), contradicting the fact that L is its limit.
         Case $L = B$: This is the same proof as the previous case; simply switch the roles of B and A and the roles of (y_n) and (x_n).
         Case $L \neq A$ and $L \neq B$: The triangle inequality tells us that $10\epsilon = |A - B| \leq |A - L| + |L - B|$. In particular, at least one of $|A - L|$ and $|L - B|$ must be greater than 2ϵ. Therefore, either the ϵ-neighborhoods of L and A are disjoint, or those of L and B are disjoint. Armed with this fact, the proofs of the previous two cases apply here, as well.
         Alternatively, if in this case you choose ϵ small enough, say $\epsilon = \min\left\{\frac{|B-A|}{10}, \frac{|B-L|}{10}, \frac{|A-L|}{10}\right\}$. Then each of ϵ-neighborhoods around A, B, and L, will be disjoint from the other two. As a tail of $\{x_n\}$ must be contained in the neighborhood around A and a tail of $\{y_n\}$ must be contained in the neighborhood around B, there exists a tail of $\{a_k\}$ that is totally disjoint from the neighborhood of L, giving us our contradiction.

ii. (A different case by case analysis)
Case where the ϵ-neighborhood of L does not contain any tail of (x_n): Since every tail of (a_k) contains a tail of (x_n), the ϵ-neighborhood of L contains no tail of (a_k), either. This contradicts the fact that L is the limit of (a_k).
Case where the ϵ-neighborhood of L contains the N-tail of (x_n): Since (a_k) converges to L, we can find a K-tail of (a_k) inside the ϵ-neighborhood of L. Since $x_n = a_{2n-1}$ and $n \geq 2n-1$, take n larger than both N and K. The triangle inequality tells us that

$$\begin{aligned} |L - A| &\leq |L - x_n| + |x_n - A| \\ &= |L - a_{2n-1}| + |x_n - A| \,. \end{aligned}$$

Since a_{2n-1} is in the K-tail of (a_k) and x_n is in the N-tail of (x_n), we know both of these distances are smaller than ϵ. Hence,

$$|L - A| < 2\epsilon.$$

Now we apply the triangle inequality again: $10\epsilon = |B - A| \leq |B - L| + |L - A| < |B - L| + 2\epsilon$. So $8\epsilon < |B - L|$. It follows that the ϵ-neighborhoods of B and L are disjoint. We can find an M-tail of (y_m) that is contained inside the ϵ-neighborhood of B. Choose m larger than both M and K. Then $|B - y_m| < \epsilon$ and $|L - y_m| = |L - a_{2m}| < \epsilon$ since $2m > m > K$. The triangle inequality gives us our contradiction:

$$8\epsilon < |B - L| \leq |B - y_m| + |y_m - L| < 2\epsilon.$$

iii. (No cases.)
We can find N_1 such that if $n > N_1$, then $|x_n - A| < \epsilon$.
We can find N_2 such that if $n > N_2$, then $|y_n - B| < \epsilon$.
We can find N_3 such that if $k > N_3$, then $|a_k - L| < \epsilon$.
Let n be larger than N_1, N_2 and N_3. By the triangle inequality:

$$\begin{aligned} 10\epsilon = |B - A| &= |B - y_n + y_n - L + L - x_n + x_n - A| \\ &\leq |B - y_n| + |y_n - L| + |L - x_n| + |x_n - A| \\ &= |B - y_n| + |a_{2n} - L| + |L - a_{2n-1}| + |x_n - A| \,. \end{aligned}$$

The fact that $n > N_1$ implies that the fourth absolute value is bounded by ϵ, while the fact that $n > N_2$ says the same about the first. The fact that $2n > 2n - 1 \geq n > N_3$ proves that each of the middle absolute values is less than ϵ, and therefore, we arrive at the contradiction:

$$10\epsilon < 4\epsilon.$$