1. (a) Prove that a constant function is differentiable at any point. Find its derivative.

Let \( f(x) = c \) for all \( x \). Let \( a \in \mathbb{R} \). For \( x \neq a \),
\[
\frac{f(x) - f(a)}{x - a} = \frac{c - c}{x - a} = 0.
\]
Therefore, \( f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) exists, and is equal to 0.

(b) Suppose that \( f \) is differentiable at \( a \). Arguing from definitions, prove that the function \( g(x) = 3f(x) + 2 \) is differentiable at \( a \).

For \( x \neq a \),
\[
\frac{g(x) - g(a)}{x - a} = 3 \frac{f(x) - f(a)}{x - a}.
\]
Since \( f'(a) \) exists, and so \( \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = 3 \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 3f'(a) \). \( g \) is differentiable at \( a \).

2. (a) Suppose that \( g(x) \) is differentiable at \( a \), and \( g(a) \neq 0 \). Arguing from definitions, prove that the function \( h(x) = \frac{1}{g(x)} \) is differentiable at \( a \), and find its derivative.

Since \( g \) is differentiable at \( a \), it is also continuous at \( a \). Therefore, since \( g(a) \neq 0 \), there is a neighborhood around \( a \) where \( g \) is never 0. For \( x \neq a \) in this neighborhood,
\[
\frac{h(x) - h(a)}{x - a} = \frac{1}{x - a} \left( \frac{1}{g(x)} - \frac{1}{g(a)} \right) = \frac{1}{x - a} \left( \frac{g(a) - g(x)}{g(a)g(x)} \right) = -\frac{g(x) - g(a)}{x - a} \frac{1}{g(a)g(x)}.
\]
Since \( g \) is continuous at \( a \) and \( g(a) \neq 0 \), \( \lim_{x \to a} \frac{1}{g(a)g(x)} = \frac{1}{g(a)^2} \). Also, since \( g \) is differentiable at \( a \), \( \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \) exists. Therefore,
\[
\lim_{x \to a} \frac{h(x) - h(a)}{x - a} = -\frac{g'(a)}{g(a)^2}
\]
is well-defined.

(b) Using part (a) and the product rule, prove the quotient rule: if \( f \), \( g \) are differentiable at \( a \), then \( \frac{f}{g} \) is also differentiable at \( a \), and
\[
\left( \frac{f}{g} \right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.
\]
Let \( h = \frac{1}{g} \), as in part (a), so that \( \frac{f}{g} = fh \). Then applying the product rule, followed by the result of part (a) yields:
\[
(fh)'(a) = f'(a)h(a) + f(a)h'(a)
= f'(a) \left( \frac{1}{g(a)} \right) + f(a) \left( -\frac{g'(a)}{(g(a))^2} \right)
= \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.
\]
3. (a) Using the product rule and induction, show that \((x^n)' = nx^{n-1}\) for all natural \(n\). 
First, for the base case, take \(n = 1\). The derivative of \(x\) at a point \(a\) is given by 
\[
\lim_{x \to a} \frac{x - a}{x - a} = 1,
\]
so \(x' = 1\), as desired. 
Now assume that for some natural number \(k\), \((x^k)' = kx^{k-1}\). We use the product rule and the base case to take the derivative of \(x^{k+1}\):
\[
(x^{k+1})' = (x^k x)' = (x^k)' x + x^k x' = kx^{k-1} x + x^k 1 = kx^k + x^k = (k + 1)x^k
\]
as desired. Therefore, the claim is proved by induction.

(b) Using question 2, find (with proof) \((x^{-5})'\).
\(x^5\) has derivative \(5x^4\). By question 2(a), \(\frac{1}{x}\) is differentiable when \(x \neq 0\), and
\[
(x^{-5})' = -\frac{5x^4}{(x^5)^2} = -\frac{5x^4}{x^{10}} = -5x^{-6}.
\]
When \(x = 0\), \(x^{-5}\) is undefined, and so it is not differentiable there.

4. (28.7) Let \(f(x) = x^2\) for \(x \geq 0\) and \(f(x) = 0\) for \(x < 0\).

(a) Sketch the graph of \(f\).

(b) Show that \(f\) is differentiable at \(x = 0\).
For \(x < 0\), \(\frac{f(x) - f(0)}{x - 0} = \frac{0}{x} = 0\). Therefore,
\[
\lim_{x \to 0^{-}} \frac{f(x) - f(0)}{x - 0} = 0.
\]
For \(x > 0\), \(\frac{f(x) - f(0)}{x - 0} = \frac{x^2}{x} = x\). Therefore,
\[
\lim_{x \to 0^{+}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^{+}} x = 0.
\]
Therefore, \(\lim_{x \to 0} f(x) = 0\). \(f'(0) = 0\).

(c) Calculate \(f'\) on \(\mathbb{R}\) and sketch its graph.
If \(a < 0\), we can find a \(\delta\)-neighborhood of \(a\), consisting entirely of negative numbers. Take \(\delta = -a > 0\), for instance. On this neighborhood, \(f \equiv 0\). Restricting our attention to this neighborhood, we can calculate:
\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{0}{x - a} = 0.
\]
If \(a > 0\), we can find a \(\delta\)-neighborhood of \(a\), consisting entirely of positive numbers. On this neighborhood, \(f(x) = x^2\). Restricting our attention to this neighborhood, we can calculate:
\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} x + a = 2a.
\]
Therefore,

\[ f'(x) = \begin{cases} 
2x & \text{if } x \geq 0 \\
0 & \text{if } x \leq 0.
\end{cases} \]

(Like continuity, differentiability is a local property. Whether or not a function is differentiable at a point is only dependent on the behavior in a small neighborhood around the point.)

(d) Is \( f' \) continuous on \( \mathbb{R} \)? differentiable on \( \mathbb{R} \)?

\( f' \) is continuous on \( \mathbb{R} \). Following question 1 in homework 8, \( f' \) is gotten by gluing together two continuous function at \( x = 0 \), and both functions agree at this point, so \( f' \) is continuous.

\( f' \) is differentiable at any point \( a \neq 0 \). (\( f \) is equivalent to a differentiable function in a neighborhood of these points. See the comment at the end of the last problem.) \( f' \) is not differentiable, however, at \( a = 0 \). To see this, note that

\[ \lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^-} \frac{0}{x} = 0 \]

while

\[ \lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^+} \frac{2x}{x} = 2; \]

so \( \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} \) is undefined.

5. (28.8) Let

\[ f(x) = \begin{cases} 
x^2 & \text{if } x \in \mathbb{Q} \\
0 & \text{if } x \notin \mathbb{Q}.
\end{cases} \]

(a) Prove that \( f \) is continuous at \( x = 0 \).

Let \( \epsilon > 0 \). Set \( \delta = \sqrt{\epsilon} \). Suppose \( |x| < \delta \). If \( x \notin \mathbb{Q} \), then \( |f(x)| = 0 < \epsilon \). If \( x \in \mathbb{Q} \), then \( |f(x)| = |x^2| < \delta^2 < \epsilon \). In either case, \( |f(x)| < \epsilon \), so \( f \) is continuous at 0.

(b) Prove that \( f \) is discontinuous at all \( x \neq 0 \).

As usual, I will prove discontinuity using the sequences definition. Let \( (r_n) \) be a sequence of rationals converging to \( x \), and let \( (s_n) \) be a sequence of irrationals converging to \( x \). Then \( (f(r_n)) = (r_n^2) \to x^2 \), while \( (f(s_n)) = (0) \to 0 \). Since \( x^2 \neq 0 \), \( f \) is discontinuous at \( x \).

(c) Prove that \( f \) is differentiable at \( x = 0 \).

We must prove that \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \) exists. We will prove that it equals 0. Let \( \epsilon > 0 \). Set \( \delta = \epsilon \). Assume \( 0 < |x| < \delta \). If \( x \in \mathbb{Q} \), then \( \left| \frac{f(x)}{x} \right| = \left| \frac{x^2}{x} \right| = |x| < \delta = \epsilon \). If \( x \notin \mathbb{Q} \), then \( \left| \frac{f(x)}{x} \right| = \left| \frac{0}{x} \right| = 0 < \epsilon \). This proves our claim.