MAT 311 Introduction to Number Theory

Problem Set 1

Solutions to a few questions

Problem 2. Prove that for every integer n

(a) $n^2 - n$ is divisible by 2

(b) $n^3 - n$ is divisible by 6

(c) $n^2 + 2$ is not divisible by 4

Solution. (a) Observe that $n^2 - n = n(n-1)$ and note that one of the numbers n and $n-1$ is always even, thus the product is even.

(b) Observe that $n^3 - n = n(n^2 - 1) = (n-1)n(n+1)$. Again, either n is even or $n-1$ (and $n+1$) is even, so the product is even. Now, n can be written as $n = 3k$, $n = 3k+1$ or $n = 3k+2$; it follows that 3 divides one of the numbers $n-1$, n or $n+1$. Since $(n-1)n(n+1)$ is divisible by 2 and 3, it is divisible by 6 (why?)

(c) Consider two cases, n odd and n even, ie $n = 2k$ or $n = 2k+1$. In the first case $n^2 + 2 = 4k^2 + 2$, in the second $n^2 + 2 = (2k+1) + 2 = 4k^2 + 4k + 3$, and neither is divisible by 4.

Problem 4. Prove that $(a, a+2) = 1$ or 2 for every integer a.

Solution. For the shortest proof, use the property $gcd(a, b) = gcd(a, b - an)$ for any integer n. Then $gcd(a, a+2) = gcd(a, a+2-a) = gcd(a, 2)$ cannot be greater than 2. (It equals to 2 if a even, 1 if a is odd).

Problem 6.

(a) Show that there are no prime triplets, that is primes p, $p+2$, $p+4$, other than 3,5, and 7.

(b) Find all prime p such that $p^2 + 1$ is also prime.

(c) Find all prime p such that $p^2 + 2$ is also prime.

Solution. (a) Unless 3 divides p (in which case $p = 3$, $p+2 = 5$, $p+4 = 7$ are all prime), we have $p = 3k+1$ or $p = 3k+2$. In the first case $p + 2 = 3k + 3$ is divisible by 3 and cannot be prime since $k > 0$, in the second case $p + 4 = 3k + 6 > 3$ is divisible by 3 and cannot be prime.

(b) Most prime numbers are odd. If p is odd, p^2 is also odd, and then $p^2 + 1$ is even. Because $p > 1$, $p^2 + 1 > 2$, and an even number greater than 2 cannot be prime. So p has to be even, then $p = 2$ (and $p^2 + 1 = 5$ is also prime).

(c) Unless $p = 3$ (which gives $p^2 + 2 = 11$, also a prime), p is not divisible by 3, so $p = 3k+1$ or $p = 3k+2$. In the first case $p^2 + 2 = 9k^2 + 6k + 1 + 2 = 9k^2 + 6k + 3 = 3(3k^2 + 2k + 1)$, in the second $p^2 + 2 = 9k^2 + 12k + 4 + 2 = 9k^2 + 12k + 6 = 3(3k^2 + 4k + 2)$, both are divisible by 3 and can't be prime (since they are greater than 3).