Tritangent circles to a generic curve

September 22, 2015
Pre-history

For a circle generically immersed to the plane
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$\#(\overbrace{\quad\quad\quad}^{\text{a}}) - \#(\overbrace{\quad\quad\quad}^{\text{a}}) = \#(\overbrace{\quad\quad\quad}^{\text{a}}) + \frac{1}{2}\#(\overbrace{\quad\quad\quad}^{\text{a}})$
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For a circle generically immersed to the plane

\[
\text{#}(\text{---}) - \text{#}(\text{---}) = \text{#}(\text{--}) + \frac{1}{2} \text{#}(\text{--})
\]

Fabricius-Bjerre formula (1962):

\[
e - i = n + \frac{1}{2} f
\]
Pre-history

For a circle generically immersed to the plane

\[
\#(\xrightarrow{\text{triangle}}) - \#(\xrightarrow{\text{disk}}) = \#(\xrightarrow{\text{circle}}) + \frac{1}{2}\#(\xrightarrow{\text{point}})
\]

Fabricius-Bjerre formula (1962):

\[
e - i = n + \frac{1}{2}f
\]

Ferrand splitted this formula (1997):

\[
e^+ - i^+ = J^+ + w^2 - 1 + \frac{1}{2}f
\]

\[
e^- - i^- = -J^- - w^2 + 1,
\]

where \( w \) is the Whitney number and \( J^\pm \) are the Arnold invariants.
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For a circle generically immersed to the plane

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where \(w\) is the Whitney number and \(J^\pm\) are the Arnold invariants.

In pictograms:

\[\#(\overrightarrow{\text{---}}) - \#(\overrightarrow{\text{---}}) = J^+ + w^2 - 1 + \frac{1}{2}\#(\overrightarrow{\text{---}})\]

\[\#(\overrightarrow{\text{---}}) - \#(\overrightarrow{\text{---}}) + \#(\overrightarrow{\text{---}}) - \#(\overrightarrow{\text{---}}) = -J^- - w^2 + 1\]
Winding numbers

Winding numbers of faces: \( f \mapsto i_f \in \mathbb{Z} \)
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The harmonic extension for a function of faces.
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The harmonic extension for a function of faces.

\[
\begin{align*}
  j + 1 & \quad j \\
  i_f = j & \quad j - 1
\end{align*}
\]

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Winding numbers

Winding numbers of faces: \( f \mapsto i_f \in \mathbb{Z} \)

\[
\begin{array}{cccc}
0 & 1 & 2 & 1 \\
1 & 0 & 1 & -1 \\
1 & 0 & -1 & 1 \\
1 & -1 & 1 & 0 \\
-1 & 1 & 0 & 1 \\
1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 \\
-1 & 1 & 0 & 1 \\
1 & -1 & 0 & 1 \\
1 & -1 & 0 & 1 \\
\end{array}
\]

The harmonic extension for a function of faces.

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$$w = \sum_f i_f − \sum_v i_v$$

$$e^+ − i^+ = \sum_f i_f^2 − \sum_v (1 + i_v^2) + w^2 + \frac{1}{2} f$$

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e^+ - i^+ = \sum_f i_f^2 - \sum_v (1 + i_v^2) + w^2 + \frac{1}{2} f
\]

\[
e^- - i^- = \sum_v i_v^2 - \sum_f i_f^2 - w^2
\]

Extra splitting of $e^\pm$, $i^\pm$, $J^\pm$ and $n$. 
Planar circles

The space of circles on $\mathbb{R}^2$
Planar circles

\[ \{ \text{The space of circles on } \mathbb{R}^2 \} = \mathbb{R}^3_0 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}. \]
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Circles tangent to a fixed line at a fixed point form two rays:
Planar circles

\[ \{ \text{The space of circles on } \mathbb{R}^2 \} = \mathbb{R}^3_0 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}. \]

Circles tangent to a fixed line at a fixed point form two rays:

Circles tangent to a curve form a surface:
Circles tangent to a curve

Let $\gamma$ be a generic immersion of $S^1$ to $\mathbb{R}^2$ or $S^2$. 
Circles tangent to a curve

Let $\gamma$ be a generic immersion of $S^1$ to $\mathbb{R}^2$ or $S^2$.

The surface $T_1$ of all circles tangent to $\gamma$, a big wave front in $\mathbb{R}_{>0}^3$. 
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Tri-tangent circles form a finite set $T_{1,1,1}$

of triple transversal self-intersections points.
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Osculating circles that are tangent \( \gamma \) at another point form a finite set \( T_{2,1} \).
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Osculating circles at extremal points of the curvature of $\gamma$ form a finite set $T_3$ of swallow tail singularities.
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Resolution of its multi-singularities

\[
S = \{ (c, p) \mid p \in S^1, c \text{ is tangent to } \gamma \text{ at } \gamma(p) \}
\]
Ordinary tritangent circles
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Cyclic order of tangency points on $\gamma$ defines the orientation of a tritangent circle $C \in T_{1,1,1}$. 
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At an ordinary tangency point, a curve is either on the right, or on the left side of the circle.
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The sign $\sigma(C')$ of the circle $C'$ is negative if the curve is on the right of the circle at odd number of points (1 or 3).
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On the picture, $\sigma = -1$. 
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A coherency of $C$ is the number of tangency points,
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A coherency of $C$ is the number of tangency points, where the orientations of $C$ and $\gamma$ agree. On the picture, the coherency is two.

Denote by $T^i$ the set of tritangent circles with coherency $i$ and put $t^i = \sum_{C \in T^i} \sigma(C)$.
Osculating tritangent circles
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Orientation of $\gamma$ at point of osculating tangency
defines the orientation of osculating tritangent circle $C \in T_{2,1}$. 
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The sign $\sigma(C')$ is negative
if the curve at the non-osculating tangency point is on the left.
Osculating tritangent circles

Orientation of $\gamma$ at point of osculating tangency defines the orientation of osculating tritangent circle $C \in T_{2,1}$.

The sign $\sigma(C)$ is negative if the curve at the non-osculating tangency point is on the left.

Denote the set of osculating tritangent circles with coherent/incoherent tangency at non-osculating point by $S^+ / S^-$, respectively.
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Let $s^\pm = \sum_{C \in S^\pm} \sigma(C)$. 
The main formulations
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Theorem (Yu. Sobolev). The numbers $t^0$, $t^1$, $\tau^2 = t^2 + \frac{s^-}{2}$ and $\tau^3 = t^3 + \frac{s^+}{2}$ are diffeomorphism invariants of $\gamma$. They change under the moves (perestroikas) of $C'$ as follows:
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<table>
<thead>
<tr>
<th></th>
<th>$\Delta(t^0)$</th>
<th>$\Delta(t^1)$</th>
<th>$\Delta(\tau^2)$</th>
<th>$\Delta(\tau^3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triple point proper strong</td>
<td>-1</td>
<td>-3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Triple point reflected strong</td>
<td>1</td>
<td>3</td>
<td>-3</td>
<td>-1</td>
</tr>
<tr>
<td>Triple point proper weak</td>
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<td>-1</td>
<td>1</td>
<td>-1</td>
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</tr>
<tr>
<td>Direct self-tangency</td>
<td>2 ind</td>
<td>-2 ind</td>
<td>2 ind</td>
<td>-2 ind</td>
</tr>
<tr>
<td>Indirect left self-tangency</td>
<td>0</td>
<td>4 ind -4</td>
<td>-4 ind +4</td>
<td>0</td>
</tr>
<tr>
<td>Indirect right self-tangency</td>
<td>0</td>
<td>4 ind +4</td>
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Formulas
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$t^0 = -\tau^3 = -\frac{1}{3}F + \frac{2}{3}E - V$ and $t^1 = -\tau^2 = F - \frac{2}{3}E + \frac{1}{3}V$